1. Find the linearization of $f(x) = \sqrt[3]{27+x}$ at $x = 0$. Use this to approximate the cube root of 27.1.

Setting $u = 27 + x$ and using the chain rule,

$$\begin{aligned}\frac{df}{dx} &= \frac{du^{1/3}}{du}\frac{d(27+x)}{dx} \\ &= (1/3)u^{-2/3}\left(\frac{d27}{dx} + \frac{dx}{dx}\right) \\ &= (1/3)(27+x)^{-2/3}(0+1).\end{aligned}$$

Thus, $f'(0) = (1/3)27^{-2/3} = 1/27$. Also, $f(0) = 27^{1/3} = 3$. The linearization at $x = 0$ is

$$L(x) = f'(0)(x-0) + f(0) = (1/27)(x-0) + 3.$$

Using the linearization to approximate the function, we have

$$\sqrt[3]{27.1} = f(.1) \approx L(.1) = (1/27)(1/10) + 3.$$

2. Find the critical points of $f(x) = x^3 - 12x + 5$ and determine the absolute maximum and minimum of $f(x)$ on $[-3, 5]$.

Using the sum/difference rule and the constant multiple rule,

$$\begin{aligned}\frac{df}{dx} &= \frac{dx^3}{dx} - \frac{d12x}{dx} + \frac{d5}{dx} \\ &= 3x^2 - 12\frac{dx}{dx} + 0 \\ &= 3x^2 - 12 = 3(x^2-4) = 3(x-2)(x+2).\end{aligned}$$

The critical points are ± 2. Note that both are in the interval we're interested in. Plugging in the critical points and endpoints of the interval gives $f(-3) = 14$, $f(-2) = 21$, $f(2) = -11$ and $f(5) = 70$. So the absolute minimum is -11 (at $x = 2$) and the absolute maximum is 70 (at $x = 5$).

3. Show that $f(x) = 2x - \cos^2 x + \sqrt{2}$ has exactly one zero.

First we need to show that there is a zero. For example, $f(-\pi/2) = -\pi + \sqrt{2} < 0$ and $f(0) = -1 + \sqrt{2} > 0$. Since f is certainly continuous, the Intermediate Value Theorem implies that f has a zero (between $-\pi/2$ and 0, in fact).

Next, using the sum/difference and constant multiple rules, and also the chain rule with $u = \cos x$, we have

$$\begin{aligned} f'(x) &= \frac{d2x}{dx} - \frac{d\cos^2 x}{dx} + \frac{d\sqrt{2}}{dx} \\ &= 2\frac{dx}{dx} - \frac{du^2}{du}\frac{d\cos x}{dx} + 0 \\ &= 2 - 2u(-\sin x) \\ &= 2(1 + \cos x \sin x). \end{aligned}$$

Since $-1 \leq \cos x, \sin x \leq 1$ for all x, and when either is ± 1 the other is 0, we clearly have $\cos x \sin x > -1$ for all x. This implies that $f'(x) > 0$ for all x.

Now if f had two zeros, say a and b. Then by the Mean Value Theorem we'd have

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0$$

for some c. Since this can never happen, f has only one zero.

4. Find an equation for the tangent line to the curve $\tan(xy) = y - (1 + x^2)^{-2}$ at the point $x = 0,\ y = 1$.

Setting $u = xy$ and $v = 1 + x^2$ and differentiating with respect to x:

$$\begin{aligned} \frac{d\tan u}{dx} &= \frac{dy}{dx} - \frac{dv^{-2}}{dx} \\ \frac{d\tan u}{du}\frac{dxy}{dx} &= y' - \frac{dv^{-2}}{dv}\frac{d1 + x^2}{dx} \\ \sec^2 u \left(y\frac{dx}{dx} + x\frac{dy}{dx} \right) &= y' + 2v^{-3}\left(\frac{d1}{dx} + \frac{dx^2}{dx} \right) \\ \sec^2(xy)(y + xy') &= y' + 2(1 + x^2)^{-3}(0 + 2x). \end{aligned}$$

Plugging in $x = 0$ and $y = 1$, we have

$$\sec^2 0(1 + 0) = y' + 2(1)^{-3}(0)$$

$$1 = y'.$$

So the slope of the tangent line is 1. In point-slope form, an equation for the tangent line is $(y-1) = 1(x-0)$.

5. A kite 80 feet above the ground moves horizontally as the string is let out at 8 feet per second. At what rate is the angle between the string and the ground changing when 100 feet of string have been let out? Give a diagram of the situation.

Your diagram should be a right triangle. The height is 80; label the hypotenuse s and the angle θ. Then $\sin\theta = 80/s$ and $\frac{ds}{dt} = 8$ (in feet per second). Differentiating our equation with respect to t gives

$$\begin{aligned} \frac{d\sin\theta}{dt} &= \frac{d80s^{-1}}{dt} \\ \frac{d\sin\theta}{d\theta}\frac{d\theta}{dt} &= 80\frac{ds^{-1}}{dt} \\ \cos\theta\frac{d\theta}{dt} &= 80\frac{ds^{-1}}{ds}\frac{ds}{dt} = -80s^{-2}(8). \end{aligned}$$

Now when $s = 100$ we have $\sin\theta = 80/100 = 4/5$. Since θ is an acute angle, using $\sin^2\theta + \cos^2\theta = 1$ gives $\cos\theta = 3/5$. Thus, when $s = 100$ we have

$$(3/5)\frac{d\theta}{dt} = \frac{-640}{100^2}$$

$$\frac{d\theta}{dt} = \frac{-5(640)}{3(100)^2}.$$

(This would be in radians per second.)

6. Compute the following limits.

a. $\lim_{x\to 0} \sec^2 x + \dfrac{2x}{\sin x}$

Using rules for limits,

$$\begin{aligned}\lim_{x\to 0} \sec^2 x + \frac{2x}{\sin x} &= \lim_{x\to 0} \sec^2 x + 2\lim_{x\to 0} \frac{x}{\sin x} \\ &= \lim_{x\to 0} \sec^2 x + \frac{2}{\lim_{x\to 0} \frac{\sin x}{x}}.\end{aligned}$$

Since the trigonometric functions are continuous on their respective domains, $\lim_{x\to 0} \sec^2 x = \sec^2 0 = 1$. Also, $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Thus,

$$\lim_{x\to 0} \sec^2 x + \frac{2x}{\sin x} = 1 + \frac{2}{1} = 3.$$

b. $\lim_{h\to 0} \dfrac{\sin^2(\pi/2 + h) - 1}{h}$

If $f(x) = \sin^2 x$, then

$$f'(\pi/2) = \lim_{h\to 0} \frac{f(\pi/2 + h) - f(\pi/2)}{h} = \lim_{h\to 0} \frac{\sin^2(\pi/2 + h) - 1}{h}.$$

On the other hand, using the chain rule with $u = \sin x$ gives

$$f'(x) = \frac{du^2}{du}\frac{d\sin x}{dx} = 2u\cos x = 2\sin x\cos x$$

so $f'(\pi/2) = 2\sin(\pi/2)\cos(\pi/2) = 0$.