

Practice Problems for Final Exam, Part II

1. A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 feet above the bow at a rate of 2 feet per second. At what rate is the boat approaching the dock when 10 feet of rope are out?

2. If $y' = x^2(x - 2)$, find all critical points, local extrema, points of inflection, intervals where y is increasing/decreasing and intervals where y is concave up/down. Use this information to make a rough sketch of the graph of y .

3. The height of a ball moving vertically is given by $s(t) = -16t^2 + 96t + 112$.

a) Find the ball's velocity.

b) Find the maximum height the ball reaches and the time at which this occurs.

c) How fast is the ball going when it hits the ground?

4. Use Newton's Method to approximate $\sqrt[4]{15}$ using two steps.

5. Compute the general anti-derivative of the following functions, showing all steps.

a) $f(x) = (x + 1)^2(x - 2) + 3x^2$

b) $g(x) = x^2 \sec(x^3) \tan(x^3)$

6. Find the upper sum for $f(x) = x^2 + 2x$ on $[2, 4]$ using 6 rectangles, then find a general formula for n rectangles. Use this formula to find the limit (definite integral).

Solution: The total length of the interval is $4 - 2 = 2$, so using n rectangles of equal width would give each of width $2/n$. Since $f(x)$ is increasing, you'll get an upper sum by using right-hand endpoints. For n rectangles, these endpoints are

$$2 + 1 \cdot \frac{2}{n}, 2 + 2 \cdot \frac{2}{n}, \dots, 2 + n \cdot \frac{2}{n}.$$

These are plugged into $f(x)$ to get the heights of the rectangles.

We need a cleaned up version of the heights to get a nice formula for the upper sum. The i^{th}

endpoint is $2 + i \cdot \frac{2}{n}$, so plugging that into $f(x)$ gives the height

$$\left(2 + \frac{2i}{n}\right)^2 + 2\left(2 + \frac{2i}{n}\right) = 4 + \frac{8i}{n} + \frac{4i^2}{n^2} + 4 + \frac{4i}{n} = 8 + \frac{12i}{n} + \frac{4i^2}{n^2}.$$

Again, each rectangle has width $2/n$. So the upper sum with n rectangles is

$$\frac{2}{n} \sum_{i=1}^n 8 + \frac{12i}{n} + \frac{4i^2}{n^2}.$$

To evaluate this sum and take the limit, we need to recall some formulas from the textbook:

$$\sum_{i=1}^n 1 = n, \quad \sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{(2n+1)(n+1)n}{6}.$$

Using the rules for sums together with these formulas, we see that the upper sum with n rectangles is

$$\begin{aligned} \frac{2}{n} \sum_{i=1}^n 8 + \frac{12i}{n} + \frac{4i^2}{n^2} &= \frac{2}{n} \left(\sum_{i=1}^n 8 + \sum_{i=1}^n \frac{12i}{n} + \sum_{i=1}^n \frac{4i^2}{n^2} \right) \\ &= \frac{2}{n} \left(8 \sum_{i=1}^n 1 + \frac{12}{n} \sum_{i=1}^n i + \frac{4}{n^2} \sum_{i=1}^n i^2 \right) \\ &= \frac{2}{n} \left(8n + \frac{12}{n} \frac{(n+1)n}{2} + \frac{4}{n^2} \frac{(2n+1)(n+1)n}{6} \right) \\ &= 16 + \frac{12(n+1)}{n} + \frac{4(2n+1)(n+1)}{3n^2} \\ &= 16 + 12 + \frac{12}{n} + \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}. \end{aligned}$$

As $n \rightarrow \infty$, these sums go to $28 + \frac{8}{3} = 92/3$.

We can check this. Since the limit should be the definite integral, we check the answer by using the Fundamental Theorem of Calculus and rules/formulas for anti-derivatives:

$$\begin{aligned} \int_2^4 x^2 + 2x \, dx &= \int_2^4 x^2 \, dx + \int_2^4 2x \, dx \\ &= \frac{x^3}{3} + x^2 \Big|_{x=2}^4 \\ &= \frac{4^3}{3} + 16 - \left(\frac{8}{3} + 4 \right) \\ &= \frac{92}{3}. \end{aligned}$$

7. Find the area of the region enclosed by the curves $y = x^4 - 2x^2$ and $y = 2x^2$.