Exercise #4 from section 4.6

We are to graph the function $y = \dfrac{x^2-1}{40x^3+x+1}$, finding all the interesting things (local extrema, concavity, etc...) via calculus *and* a calculator/computer.

First, this is neither an even nor an odd function. Next, we verify that $y = 0$ is a horizontal asymptote since

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{x^2-1}{40x^3+x+1} &= \lim_{x\to\pm\infty} \frac{x^{-1}-x^{-3}}{40+x^{-2}+x^{-3}} \\ &= \frac{0-0}{40+0+0} \\ &= 0.\end{aligned}$$

As for vertical asymptotes, we need to find the roots of the denominator $40x^3+x+1$. Call this function f. Then $f(-1) = -40-1+1 = -40 < 0$ and $f(0) = 0+0+1 = 1 > 0$. So by the Intermediate Value Theorem, f has a root between -1 and 0. On the other hand,

$$f'(x) = \frac{d40x^3}{dx} + \frac{dx}{dx} + \frac{d1}{dx} = 40\frac{dx^3}{dx} + 1 = 120x^2 + 1 \geq 1 > 0,$$

so by the Mean Value Theorem (or just Rolle's Theorem), f can have at most one root. Using your calculator, you can locate the one root of f; it's approximately $-.264$. So we have one vertical asymptote located pretty close to $x = -.264$.

We now turn to local extrema. For that, we need the first derivative. Note that we computed the derivative of the denominator $f(x)$ above. The derivative of the numerator (call it g) is

$$\frac{dg}{dx} = \frac{dx^2}{dx} - \frac{d1}{dx} = 2x.$$

Using the quotient rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dg}{dx}f(x) - g(x)\frac{df}{dx}}{\big(f(x)\big)^2} \\ &= \frac{2x(40x^3+x+1) - (x^2-1)(120x^2+1)}{(40x^3+x+1)^2} \\ &= \frac{80x^4+2x^2+2x-(120x^4-119x^2-1)}{(40x^3+x+1)^2} \\ &= \frac{-40x^4+121x^2+2x+1}{(40x^3+x+1)^2}.\end{aligned}$$

The root of the denominator will not be a critical number, since it isn't in the domain of the original function. Thus, the only critical numbers are roots of the numerator. We discussed in class what the general shape of the graph of the numerator must look like. Call the numerator here h. Since $h(0) = 1$ and $h \to -\infty$ as $x \to \pm\infty$, we know from the Intermediate Value Theorem that there are at least two roots. The question is whether that middle "hump" in its graph drops down below the x-axis. Graphing h with a window of $[-2,2]\times[-10,10]$, for example, shows that it only has two roots. You can use your calculator to locate these roots; they are approximately 1.75 and -1.75. (Note: where I goofed up in class on Wednesday was by dropping a sign and getting $-40x^4+121x^2+2x-1$ for the numerator. OOPS!) Not only that, but via the First Derivative Test, we know the negative critical number here will give us a local minimum, and the positive one will give us a local maximum (this is because the denominator in y' is a square, so always positive, therefore the sign of y' is just the sign of the numerator).

What's left is to figure out concavity and points of inflection. For that we will need the second derivative. To make that calculation neater, we'll do some preparatory work first:

$$\begin{aligned}\frac{dh}{dx} &= -\frac{d40x^4}{dx}+\frac{d121x^2}{dx}+\frac{d2x}{dx}+\frac{d1}{dx}\\ &= -40\frac{dx^4}{dx}+121\frac{dx^2}{dx}+2\frac{dx}{dx}\\ &= -160x^3+242x+2\end{aligned}$$

and

$$\frac{df^{-2}}{dx}=\frac{df^{-2}}{df}\cdot\frac{df}{dx}=-2f^{-3}\cdot(120x^2+1)=-(240x^2+2)f^{-3}.$$

Using these, we get

$$\begin{aligned}y''=\frac{dhf^{-2}}{dx} &= \frac{dh}{dx}f^{-2}+h\frac{df^{-2}}{dx}\\ &= (-160x^3+242x+2)f^{-2}-(-40x^4+121x^2+2x+1)(240x^2+2)f^{-3}\\ &= f^{-3}\big((-160x^3+242x+2)(40x^3+x+1)-(-40x^4+121x^2+2x+1)(240x^2+2)\big).\end{aligned}$$

We already know about the roots of $f(x)=40x^3+x+1$ factor, so we'll just concentrate on the rest:

$$\begin{aligned}&(-160x^3+242x+2)(40x^3+x+1)-(-40x^4+121x^2+2x+1)(240x^2+2)\\ &= -6400x^6+9520x^4-80x^3+242x^2+244x+2\\ &\qquad +9600x^6-28960x^4-480x^3-482x^2-4x-2\\ &= 3200x^6-19440x^4-560x^3-240x^2+240x\\ &= x(3200x^5-19440x^3-560x^2-240x+240).\end{aligned}$$

We have a point of inflection at $x=0$ for sure. The other points of inflection occur at the roots of the polynomial $3200x^5-19440x^3-560x^2-240x+240$. Using you calculator, you can locate the three roots of this polynomial; they are at approximately .205, 2.48 and -2.45. In summary, the graph is concave down on approximately $(-\infty,-2.45)$, $(-.264,0)$ and $(.205,2.48)$. The graph is concave up the rest of the time.