Math 229 Section 1 Quiz #7 Solutions

1. Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.

The function $f(x) = 2x - 1 - \sin x$ is continuous. Since $f(0) = -1$ and $f(2) = 3 - \sin 2 \geq 3 - 2 = 1$, the Intermediate Value Theorem implies that $f(c) = 0$ for some c in the interval $(0, 2)$. Moreover,

$$f'(x) = \frac{d2x}{dx} - \frac{d1}{dx} - \frac{d\sin x}{dx} = 2 - \cos x \geq 2 - 1 = 1,$$

so Rolle's Theorem implies that $f(x)$ cannot equal zero more than once. Thus, $f(x)$ is equal to zero exactly once.

2. Find the critical numbers of $f(x) = x^4(x-1)^3$. What does the First Derivative Test tell you?

By the product rule and the chain rule with $u = x - 1$,

$$\begin{aligned} f'(x) &= \frac{dx^4}{dx}(x-1)^3 + x^4\frac{d(x-1)^3}{dx} \\ &= 4x^3(x-1)^3 + x^4\frac{du^3}{du}\frac{d(x-1)}{dx} \\ &= 4x^3(x-1)^3 + x^4 3u^2\left(\frac{dx}{dx} - \frac{d1}{dx}\right) \\ &= 4x^3(x-1)^3 + 3x^4(x-1)^2 \\ &= x^3(x-1)^2\big(4(x-1) + 3x\big) \\ &= x^3(x-1)^2(7x-4). \end{aligned}$$

The critical numbers are 0, 1 and 4/7. We also see that f' is positive on $(-\infty, 0)$ and $(4/7, \infty)$; it is negative on $(0, 1)$ and $(1, 4/7)$. The First Derivative Test tells us that we have a local maximum at $x = 0$ and a local minimum at $x = 4/7$.

3. For $g(x) = 200 + 8x^3 + x^4$, find the intervals of increase or decrease, the local maximum and minimum values, the intervals of concavity and the inflection points.

First find $g'(x)$ and $g''(x)$:

$$\begin{aligned} g'(x) &= \frac{d200}{dx} + \frac{d8x^3}{dx} + \frac{dx^4}{dx} \\ &= 8\frac{dx^3}{dx} + 4x^3 \\ &= 24x^2 + 4x^3, \end{aligned}$$

and

$$\begin{aligned} g''(x) &= \frac{d24x^2}{dx} + \frac{d4x^3}{dx} \\ &= 24\frac{dx^2}{dx} + 4\frac{dx^3}{dx} \\ &= 48x + 12x^2. \end{aligned}$$

Factoring both of these gives

$$g'(x) = 4x^2(6 + x), \qquad g''(x) = 12x(4 + x).$$

You can check that g' is negative on $(-\infty, -6)$ and positive on $(-6, 0)$, and $(0, \infty)$. We have a local minimum at -6 and just a horizontal tangent with no local extremum at 0. You can check that g'' is negative on $(-4, 0)$ and positive on $(-\infty, -4)$ and $(0, \infty)$. The graph is concave down on $(-4, 0)$ and concave up on $(-\infty, -4)$ and $(0, \infty)$; there are points of inflection at 0 and -4.