

Math 229 Section 1 Quiz #8 Solutions

1. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.

Draw a picture of a rectangle and label the sides x and y . Then the perimeter is $2x + 2y$. The area is $xy = 1000$, so $y = 1000/x$ and the perimeter is $P(x) = 2x + 2000/x$. We're supposed to minimize $P(x)$ (of course $0 < x$). Taking derivatives,

$$P'(x) = \frac{d2x}{dx} + \frac{d2000x^{-1}}{dx} = 2\frac{dx}{dx} + 2000\frac{dx^{-1}}{dx} = 2 - 2000x^{-2}.$$

We want the critical numbers (at least the positive ones), so we factor P' , getting

$$P'(x) = 2x^{-2}(x^2 - 1000).$$

There is only one positive critical number, and that is $\sqrt{1000}$. You can check via either the First Derivative Test or find P'' and use the Second Derivative Test to see that P will be minimized at this critical number. So the perimeter is minimized when $x = \sqrt{1000}$. That gives $y = \sqrt{1000}$, too, since $y = 1000/x$.

2. Use the guidelines developed in class to sketch the curve $y = (4 - x^2)^5$.

This is a polynomial with only even powers of x , so it's an even function. There are no horizontal or vertical asymptotes. To find local extrema, take the first derivative using the chain rule with $u = 4 - x^2$. Then

$$y' = \frac{du^5}{dx} = \frac{du^5}{du} \frac{d4 - x^2}{dx} = 5u^4 \left(\frac{d4}{dx} - \frac{dx^2}{dx} \right) = 5(4 - x^2)^4(-2x).$$

The critical numbers are ± 2 and 0 . Looking at the sign of y' , you see that y is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$; there is a local maximum at $x = 0$ and just horizontal tangents (no extrema) at $x = \pm 2$.

For concavity we need the second derivative. Using the product and chain rules with the same u (recall we already computed du/dx above),

$$\begin{aligned} y'' &= \frac{d5(4 - x^2)^4}{dx}(-2x) + 5(4 - x^2)^4 \frac{d-2x}{dx} \\ &= 5 \frac{du^4}{dx}(-2x) - 10(4 - x^2)^4 \frac{dx}{dx} \\ &= -10x \frac{du^4}{du} \frac{du}{dx} - 10(4 - x^2)^4 \\ &= 80x^2 u^3 - 10u^4 \\ &= 80u^3(x^2 - u/8) \\ &= 80(4 - x^2)^3(x^2 - (4 - x^2)/8) \\ &= 80(4 - x^2)^3(7x^2/8 - 1/2) \\ &= 70(4 - x^2)^3(x^2 - 4/7). \end{aligned}$$

The function is concave down on $(-\infty, -2)$, $(-2/\sqrt{7}, 2/\sqrt{7})$ and $(2, \infty)$. It is concave up on $(-2, -2/\sqrt{7})$ and $(2/\sqrt{7}, 2)$. There are points of inflection at ± 2 and $\pm 2/\sqrt{7}$. Using this information, you should be able to come up with a nice graph.

3. Find the horizontal and vertical asymptotes of the curve $y = \frac{1 + x^4}{x^2 - x^4}$.

The vertical asymptotes occur where the denominator is 0. Factoring the denominator we have $x^2 - x^4 = x^2(1 - x^2) = x^2(1 - x)(1 + x)$, so the vertical asymptotes are $x = 0$, $x = -1$ and $x = 1$. For the horizontal asymptotes, we look at the two limits (since they're both the same, I've combined the calculations here):

$$\begin{aligned} \lim_{x \rightarrow \pm\infty} \frac{1 + x^4}{x^2 - x^4} &= \lim_{x \rightarrow \pm\infty} \frac{(1 + x^4) \times x^{-4}}{(x^2 - x^4) \times x^{-4}} \\ &= \lim_{x \rightarrow \pm\infty} \frac{x^{-4} + 1}{x^{-2} - 1} \\ &= \frac{\lim_{x \rightarrow \pm\infty} x^{-4} + 1}{\lim_{x \rightarrow \pm\infty} x^{-2} - 1} \\ &= \frac{\lim_{x \rightarrow \pm\infty} x^{-4} + \lim_{x \rightarrow \pm\infty} 1}{\lim_{x \rightarrow \pm\infty} x^{-2} - \lim_{x \rightarrow \pm\infty} 1} \\ &= \frac{0 + 1}{0 - 1} = -1. \end{aligned}$$

There is one horizontal asymptote, $y = -1$.