The Division Algorithm: For any polynomials $A(X)$ and $B(X)$ with $B(X) \neq 0$, there are polynomials $Q(X)$ and $R(X)$ with

$$A(X) = Q(X) \times B(X) + R(X)$$

and $\deg(R) < \deg(B)$.

Proof (shockingly similar to the proof in the book): If $A(X) = 0$, then we simply let $Q(X) = R(X) = 0$, too, and we're done.

Suppose $A(X) \neq 0$. Consider the set of all polynomials of the form $A(X) - Q(X)B(X)$. If 0 is in this set, then we're done. If not, then this set has an element of least degree (since the set of degrees of such polynomials is a non-empty subset of the natural numbers). Let $R(X)$ be such a polynomial. Suppose $\deg(R) \geq \deg(B)$. We can write

$$R(X) = r_m X^m + \cdots + r_0,$$

where $r_m \neq 0$. Write

$$B(X) = b_n X^n + \cdots + b_0,$$

where $b_n \neq 0$ and $n \leq m$. Then

$$R(X) - \frac{r_m}{b_n} X^{m-n} B(X) = 0X^m + \cdots$$

is also in the set of polynomials above, and moreover its degree is less than m. This contradicts the way $R(X)$ was chosen, so $\deg(R) < \deg(B)$.