Some Solutions for Week 7 Homework

13. a) Suppose τ is the k-cycle $(1, 2, \ldots, k)$ and σ is an element of S_n. Let $i \in \{1, \ldots, n\}$. Then $\sigma^{-1}(\sigma(i)) = i$ since $\sigma^{-1} \circ \sigma$ is the identity function.

If $i < k$, then $\tau(i) = i + 1$, so that

$$\sigma \circ \tau \circ \sigma^{-1}(\sigma(i)) = \sigma\big(\tau(i)\big) = \sigma(i+1).$$

If $i = k$, then $\tau(i) = \tau(k) = 1$, so that

$$\sigma \circ \tau \circ \sigma^{-1}(\sigma(k)) = \sigma\big(\tau(k)\big) = \sigma(1).$$

Finally, if $i > k$, then $\tau(i) = i$, so that

$$\sigma \circ \tau \circ \sigma^{-1}(\sigma(i)) = \sigma\big(\tau(i)\big) = \sigma(i).$$

Since σ is necessarily onto, $\{1, 2, \ldots, n\} = \{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$. Thus, we have shown what the composition $\sigma \circ \tau\sigma^{-1}$ does to each of the integers 1 through n. This shows that

$$\sigma \circ \tau \circ \sigma^{-1} = \big(\sigma(1), \sigma(2), \ldots, \sigma(k)\big).$$

Editorial comment: You weren't supposed to assume that $k = n$ for this exercise.

b) Let ρ be a k-cycle: $\rho = (i_1, i_2, \ldots, i_k)$. Then the $i_1, i_2, \ldots, i_k$ are distinct elements of $\{1, \ldots, n\}$. Thus, we can order the remaining $n - k$ elements any way we like. Choose such an ordering, so that

$$\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\},$$

and let σ be the permutation defined by $\sigma(j) = i_j$ for all $j = 1, 2, \ldots, n$. According to part a) above,

$$\sigma \circ \tau \circ \sigma^{-1} = \big(\sigma(1), \sigma(2), \ldots, \sigma(k)\big) = (i_1, i_2, \ldots, i_k) = \rho.$$

15. For $\alpha, \beta \in S_n$, say $\alpha \sim \beta$ if $\beta = \sigma \circ \alpha \circ \sigma^{-1}$ for some $\sigma \in S_n$.

Let α be any element of S_n. If σ is the identity function, then $\sigma \circ \alpha \circ \sigma^{-1} = \alpha$. So $\sim$ is reflexive.

Suppose $\alpha \sim \beta$. Then $\beta = \sigma \circ \alpha \circ \sigma^{-1}$ for some $\sigma \in S_n$. From this, we get

$$\sigma^{-1} \circ \beta \circ \sigma = \sigma^{-1} \circ \sigma \circ \alpha \circ \sigma^{-1} \circ \sigma = \alpha.$$

(Here I've used associativity of composition, the definition of inverse, and the fact that composing with the identity function leaves the original function unchanged.) We may rewrite this as

$$(\sigma^{-1}) \circ \beta \circ (\sigma^{-1})^{-1} = \alpha,$$

so $\beta \sim \alpha$.

Suppose $\alpha \sim \beta$ and $\beta \sim \gamma$. Then $\beta = \sigma \circ \alpha \circ \sigma^{-1}$ and $\gamma = \tau \circ \beta \circ \tau^{-1}$ for some $\sigma, \tau \in S_n$. This implies that

$$\gamma = \tau \circ \sigma \circ \alpha \circ \sigma^{-1} \circ \tau^{-1} = (\tau \circ \sigma) \circ \alpha \circ (\tau \circ \sigma)^{-1},$$

so that $\alpha \sim \gamma$.

Editorial comment: You'll note how I used

$$(\sigma^{-1})^{-1} = \sigma$$

and

$$(\tau \circ \sigma)^{-1} = \sigma^{-1} \circ \tau^{-1}$$

above. These two basic facts are not difficult to prove on your own. We'll see this again in the beginning of chapter 3.