CAN TWO GROUPS BE THE SAME WITHOUT BEING EQUAL?

On Wednesday we found that a group of order 6 which is not abelian must have an element of order 2 (call it f) and an element of order 3 (call it g) with $fg \neq gf$. The multiplication table is

$\times$	e	f	g	g^2	fg	gf
e	e	f	g	g^2	fg	gf
f	f	e	fg	gf	g	g^2
g	g	gf	g^2	e	f	fg
g^2	g^2	fg	e	g	gf	f
fg	fg	g^2	gf	f	e	g
gf	gf	g	f	fg	g^2	e

Note how f, fg and gf are all order 2, g and g^2 are order 3, and no element of order 2 commutes with an element of order 3.

We can apply this to the group S_3. Here we of course have $e = (1)$, but past that we have choices. For instance, we could use $f = (1, 2)$ and $g = (1, 2, 3)$. We could also have $f = (1, 3)$ and $g = (1, 2, 3)$. Really the only restriction is that f must be one of the elements of order 2 and g must be one of the elements of order 3.

Clearly the $e, f, g, \ldots$ are just *names* we chose for these elements. After all, "f" is a lot shorter (and fits in a table better) than "the function from the set consisting of the numbers 1, 2 and 3 to this set which sends 1 to 2, 2 to 1, and 3 to 3."

The invertible 2×2 matrices with entries in the integers modulo 2 are:

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} \qquad \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}$$
$$\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix} \qquad \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix} \qquad \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

As usual, the first matrix here is the identity. The second (as you can check) has order 2 and the third has order 3. They don't commute. So this group has the exact same multiplication table! We could, for example, set

$$f = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} \qquad g = \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}$$

So, while these are different groups, they really are the same. The only *real* difference is the names we choose for the elements.