Natural Numbers: Peano Arithmetic

In Math 420 we take the integers for granted. By that I mean we don't "prove" their existance nor the usual properties (e.g., addition is commutative). The purpose of this note is to describe one way - a typical way - of developing the natural numbers from sets, or to start with axioms as we did with the integers. Later we'll see how to define the integers using the natural numbers and other notions from chapter 2 of the textbook.

Axioms in this context doesn't merely mean "self-evident truth" á la the Greek geometers. Surely there is some of that (after all, everybody "knows" how natural numbers behave on a very intuitive level), but there is also a fair degree of the mathematical way of using axioms simply as a matter of defining objects of study.

At any rate, the *Peano Axioms* (actually first explicitly listed by Dedekind in the late 1800's) are typically taken as the properties we need of $\mathbb{N}$, the natural numbers 0, 1, 2, ... It's a pretty minimalist set of axioms; they are enough to get the job done, so to speak, without unnecessary bulk. Without further ado, here they are.

P1: There is a number 0.

P2: For each number n, there is a successor n' which is not equal to n.

P3: For no number n is $n' = 0$.

P4: If m and n are numbers such that $m' = n'$, then $m = n$.

P5: If S is a set of numbers such that $0 \in S$ and, for every $n \in S$, $n' \in S$, then $S = \mathbb{N}$.

The fifth axiom contains the all-important "principal of mathematical induction" found in your textbook.

You'll note that there is no mention of addition and multiplication. That's because these can be developed using the Peano axioms. Of course it's important that, while perhaps tedious to carry out, this development is possible and yields the usual stuff about associativity, commutativity, etc. Once you have addition, the ordering of $\mathbb{N}$ is simple: $a \leq b$ if $a + n = b$ for some n. You then get the expected properties of $\leq$. Once we have $\mathbb{N}$ together with addition and multiplication, getting the integers $\mathbb{Z}$ and rational numbers $\mathbb{Q}$ is an interesting application of some stuff from chapter 2 of the textbook.

Of course, one could reasonably complain that the above doesn't really tell us what the natural numbers *are*, only how they should behave. Using set theory, it's possible to construct the natural numbers (in a way that is, perhaps, not so natural). Of course, this just buries the original complaint one level deeper. ("What's a set?!?")

As a notational convenience, denote $x \cup \{x\}$ by x'. This is often called the successor set of x. By ZF8, there is a set x with

$$\emptyset, \emptyset', \emptyset'', \ldots \in x.$$

Note that we can't assert that x consists of precisely these elements and nothing more (yet). Let's say a set S is a successor set if $\emptyset \in S$ and, whenever $y \in S$, $y' \in S$ as well. Then x is a successor set.

Theorem: There is a successor set which is a subset of all successor sets.

Proof: By the power set axiom ZF5, the power set of x, $P(x)$ is a set. (Recall that $P(x)$ is the set of all subsets of x.) By ZF6, the collection of all subsets $y \in P(x)$ which are successor sets is a set. Call this collection v. Using ZF6 once more, there is a set which is the intersection of all the sets in v. Call this set ω. We have $\omega \subset y$ for all $y \in v$ by construction. It's easy to see that the intersection of successor sets is a successor set, so that $\omega \in v$, i.e., ω is a successor set.

Now suppose z is a successor set. Then $z \cap x$ is a successor set, and thus an element of v. But then $\omega \subset z \cap x$. Since $z \cap x \subset z$, this shows that $\omega \subset z$. So ω is a successor set which is a subset of all successor sets.

Our set ω can be used as $\mathbb{N}$, where we think of "0" as $\emptyset$, "1" as the successor set $\emptyset'$, and so on. It turns out that ω satisfies the Peano axioms. It even satisfies the axiom used to get induction; all the work to get that is done in the construction of ω.