The Real Numbers

Here we show one way to explicitly construct the real numbers $\mathbb{R}$. First we need a definition.

Definitions/Notation: A sequence of rational numbers is a funtion $f: \mathbb{N} \to \mathbb{Q}$. Rather than write f and $f(n)$ for the sequence and its values, we typically write $\{a_n\}$ and a_n. A sequence $\{a_n\}$ is called a Cauchy sequence if, for all $\epsilon > 0$ (which we may take to be a rational epsilon for now), there is an $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$. For any rational number c, we let "c" (with the quotes) denote the sequence $\{a_n\}$ defined by $a_n = c$ for all $n \in \mathbb{N}$. In other words, "c" is the sequence consisting entirely of the number c. This is clearly a Cauchy sequence. We say a sequence $\{a_n\}$ converges to the (rational) number c, and write $\{a_n\} \to c$, if for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - c| < \epsilon$ for all $n \geq N$.

Lemma: If $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then so are $\{a_n + b_n\}$ and $\{a_n b_n\}$. All Cauchy sequences are bounded.

Proof: Let $\epsilon > 0$. By definition, there are $N_1, N_2 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon/2$ for all $n, m \geq N_1$ and $|b_n - b_m| < \epsilon/2$ for all $n, m \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for all $n, m \geq N$ we have

$$\begin{aligned}
|(a_n + b_n) - (a_m + b_m)| &= |(a_n - a_m) + (b_n - b_m)| \\
&\leq |a_n - a_m| + |b_n - b_m| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

This shows that $\{a_n + b_n\}$ is a Cauchy sequence.

Next, get $N_3, N_4 \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N_3$ and $|b_n - b_m| < 1$ for all $n, m \geq N$. Set $B_1 = \max_{n \leq N_3}\{|a_n| + 1\}$ and $B_2 = \max_{n \leq N_4}\{|b_n| + 1\}$. One readily verifies that $|a_n| < B_1$ and $|b_n| < B_2$ for all $n \in \mathbb{N}$ (this shows that all Cauchy sequences are bounded). Note that $B_1, B_2 > 0$ by construction. Now there are $N_5, N_6 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon/(2B_2)$ for all $n, m \geq N_5$ and $|b_n - b_m| < \epsilon/(2B_1)$ for all $n, m \geq N_6$. Let $N' = \max\{N_5, N_6\}$. Then for all

$n, m \geq N'$ we have

$$\begin{aligned} |a_n b_n - a_m b_m| &= |a_n(b_n - b_m) + b_m(a_n - a_m)| \\ &\leq |a_n(b_n - b_m)| + |b_m(a_n - a_m)| \\ &= |a_n| \cdot |b_n - b_m| + |b_m| \cdot |a_n - a_m| \\ &< B_1 \cdot \epsilon/(2B_1) + B_2 \cdot \epsilon/(2B_2) \\ &= \epsilon. \end{aligned}$$

This shows that $\{a_n b_n\}$ is a Cauchy sequence.

Definition: We say two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers are equivalent, and write $\{a_n\} \sim \{b_n\}$, if the sequence $\{a_n - b_n\} \to 0$.

Lemma: This is an equivalence relation.

Proof: For any sequence $\{a_n\}$ of rational numbers, $\{a_n - a_n\} =$ "0" $\to 0$, so that $\{a_n\} \sim \{a_n\}$ for any Cauchy sequence $\{a_n\}$.

Suppose $\{a_n\} \sim \{b_n\}$ and let $\epsilon > 0$. Then for some $N \in \mathbb{N}$, $|a_n - b_n| < \epsilon$ for all $n \geq N$. Thus $|b_n - a_n| < \epsilon$ for $n \geq N$ and $\{b_n - a_n\} \to 0$. In other words, $\{b_n\} \sim \{a_n\}$.

Suppose $\{a_n\} \sim \{b_n\}$ and $\{b_n\} \sim \{c_n\}$ and let $\epsilon > 0$. Then for some $N \in \mathbb{N}$, $|a_n - b_n| < \epsilon/2$ for all $n \geq N$ and for some $M \in \mathbb{N}$, $|b_n - c_n| < \epsilon/2$ for all $n \geq M$. This implies that

$$|a_n - c_n| = |a_n - b_n + b_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq \max\{N, M\}$. Thus $\{a_n - c_n\} \to 0$ and $\{a_n\} \sim \{c_n\}$.

Definition: The real numbers $\mathbb{R}$ is the set of equivalence classes of Cauchy sequences of rational numbers. We'll write $[\{a_n\}]$ to denote the equivalence class which contains the sequence $\{a_n\}$. We view $\mathbb{Q}$ as a subset of $\mathbb{R}$ by identifying the rational number c with ["c"].

Definition: Define addition and multiplication of real numbers by

$$[\{a_n\}] + [\{b_n\}] = [\{a_n + b_n\}]$$

and

$$[\{a_n\}] \cdot [\{b_n\}] = [\{a_n \cdot b_n\}].$$

Lemma: These operations are well-defined, i.e., depend only on the equivalence classes and not the particular elements of the equivalence classes used.

Proof: Suppose $\{a_n\} \sim \{a'_n\}$ and $\{b_n\} \sim \{b'_n\}$ are all Cauchy sequences. By the lemma above, both $\{a_n + b_n\}$ and $\{a_n b_n\}$ are Cauchy sequences.

Let $\epsilon > 0$. Then for some $N_1, M_1 \in \mathbb{N}$, $|a_n - a'_n| < \epsilon/2$ for all $n \geq N_1$ and $|b_n - b'_n| < \epsilon/2$ for all $n \geq M_1$. This implies that

$$|(a_n + b_n) - (a'_n + b'_n)| = |(a_n - a'_n) + (b_n - b'_n)| \leq |a_n - a'_n| + |b_n - b'_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq \max\{N_1, M_1\}$. Thus, $\{a_n + b_n\} \sim \{a'_n + b'_n\}$ and addition is well-defined.

By a previous lemma, all four sequences $\{a_n\}, \{a'_n\}, \{b_n\}$ and $\{b'_n\}$ are bounded. Thus, there is a $B > 0$ such that $|a_n|, |a'_n|, |b_n|, |b'_n| \leq B$ for all $n \in \mathbb{N}$. For some $N_2, M_2 \in \mathbb{N}$ we have $|a_n - a'_n| < \frac{\epsilon}{2B}$ for all $n \geq N_2$ and $|b_n - b'_n| < \frac{\epsilon}{2B}$ for all $n \geq M_2$. This implies that

$$\begin{aligned}
2|a_n b_n - a'_n b'_n| &= |(a_n - a'_n)(b_n + b'_n) + (a_n + a'_n)(b_n - b'_n)| \\
&\leq |(a_n - a'_n)(b_n + b'_n)| + |(a_n + a'_n)(b_n - b'_n)| \\
&= |a_n - a'_n| \cdot |b_n + b'_n| + |a_n + a'_n| \cdot |b_n - b'_n| \\
&\leq |a_n - a'_n| \cdot (|b_n| + |b'_n|) + (|a_n| + |a'_n|) \cdot |b_n - b'_n| \\
&< \frac{\epsilon}{2B} \cdot (2B) + \frac{\epsilon}{2B} \cdot (2B) \\
&= 2\epsilon
\end{aligned}$$

for all $n \geq \max\{N_2, M_2\}$. Thus, $|a_n b_n - a'_n b'_n| < \epsilon$ for all $n \geq \max\{N_2, M_2\}$ and $\{a_n b_n\} \sim \{a'_n b'_n\}$. This shows that multiplication is well-defined.

Theorem 1: The real numbers are a field.

Proof: Since the rational numbers are a field,

$$([\{a_n\}] + [\{b_n\}]) + [\{c_n\}] = [\{(a_n + b_n) + c_n\}] = [\{a_n + (b_n + c_n)\}] = [\{a_n\}] + ([\{b_n\}] + [\{c_n\}])$$

for any Cauchy sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$, and similarly for multiplication. Also,

$$[\{a_n\}] + [\{b_n\}] = [\{a_n + b_n\}] = [\{b_n + a_n\}] = [\{b_n\}] + [\{a_n\}],$$

and similarly for multiplication. Next,

$$([\{a_n\}] + [\{b_n\}]) \cdot [\{c_n\}] = [\{(a_n + b_n) \cdot c_n\}] = [\{(a_n \cdot c_n) + (b_n \cdot c_n)\}] = ([\{a_n\}] \cdot [\{c_n\}]) + ([\{b_n\}] \cdot [\{c_n\}]).$$

Since 0 is the additive identity element of $\mathbb{Q}$,

$$[\text{“}0\text{”}] + [\{a_n\}] = [\{0 + a_n\}] = [\{a_n\}]$$

for any $[\{a_n\}] \in \mathbb{R}$, so that $[\text{“}0\text{”}]$ is an additive identity for $\mathbb{R}$. Clearly

$$[\{a_n\}] + [\{-a_n\}] = [\{a_n + (-a_n)\}] = [\text{“}0\text{”}]$$

for any $[\{a_n\}] \in \mathbb{R}$, so that $[\{-a_n\}]$ is an additive inverse for $[\{a_n\}]$.

Since 1 is the multiplicative identity element of $\mathbb{Q}$

$$[\{a_n\}] \cdot [\text{“}1\text{”}] = [\{a_n \cdot 1\}] = [\{a_n\}]$$

for all $[\{a_n\}] \in \mathbb{R}$. Clearly “0” $\not\sim$ “1”. Suppose that $[\{a_n\}] \neq [\text{“}0\text{”}]$, i.e., $\{a_n\} \not\sim$ “0”. This means that for some $\epsilon > 0$, there are infinitely many $n \in \mathbb{N}$ such that $|a_n| \geq \epsilon$. Since $\{a_n\}$ is a Cauchy sequence, there is an $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon/2$ for all $n, m \geq N$. Since there must be an $n_0 \geq N$ such that $|a_{n_0}| \geq \epsilon$, then

$$|a_m| \geq |a_{n_0}| - |a_{n_0} - a_m| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

for all $m \geq N$. In particular, $a_m \neq 0$ for $m \geq N$. Further, it is not difficult to see that the sequence $\{b_n\}$ defined by

$$b_n = \begin{cases} a_N & \text{if } n \leq N, \\ a_n & \text{if } n \geq N \end{cases}$$

is a Cauchy sequence equivalent to $\{a_n\}$. Thus, we may assume without loss of generality that $a_n \neq 0$ for all $n \in \mathbb{N}$. Now since the rational numbers are a field,

$$[\{a_n\}] \cdot [\{a_n^{-1}\}] = [\{a_n \cdot a_n^{-1}\}] = [\text{“}1\text{”}],$$

so that $[\{a_n\}]$ has a multiplicative inverse.

Definition: For $[\{a_n\}] \in \mathbb{R}$, $\big|[\{a_n\}]\big| = [\{|a_n|\}]$.

Lemma: This is well-defined, i.e., $\big|[\{a_n\}]\big| \in \mathbb{R}$ and depends only on the equivalence class of $\{a_n\}$.

Proof: Let $\epsilon > 0$. Then for some $N \in \mathbb{N}$ we have $|a_n - a_m| < \epsilon$ for all $n, m \geq N$. Since $\big||a_n| - |a_m|\big| \leq |a_n - a_m|$, this implies that $\big||a_n| - |a_m|\big| < \epsilon$ for all $n, m \geq N$ and $\{|a_n|\}$ is a Cauchy sequence.

Suppose $\{a'_n\} \sim \{a_n\}$ and let $\epsilon > 0$. Then for some $N \in \mathbb{N}$, $|a'_n - a_n| < \epsilon$ for all $n \geq N$. As above, this implies that $\big||a'_n| - |a_n|\big| < \epsilon$ for all $n \geq N$, so that $\{|a'_n|\} \sim \{|a_n|\}$.

Definition: We say a real number $[\{a_n\}]$ is greater than zero (or positive), and write $[\{a_n\}] > 0$, if there is an $N \in \mathbb{N}$ and an $\epsilon > 0$ such that $a_n \geq \epsilon$ for all $n \geq N$.

Lemma: This is well-defined.

Proof: Suppose $\{a'_n\}$ and $\{a_n\}$ are equivalent Cauchy sequences. Suppose further that there is some $N \in \mathbb{N}$ and an $\epsilon > 0$ such that $a_n \geq \epsilon$ for all $n \geq N$. There is an $M \in \mathbb{N}$ such that $|a'_n - a_n| < \epsilon/2$ for all $n \geq M$. This implies that $|a'_n| \geq |a_n| - |a'_n - a_n| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$ for all $n \geq \max\{N, M\}$.

Definition: We say a real number $[\{a_n\}]$ is greater than a real number $[\{b_n\}]$, and write $[\{a_n\}] > [\{b_n\}]$, if $[\{a_n\}] - [\{b_n\}] > 0$.

Theorem 2: For any $[\{a_n\}], [\{b_n\}] \in \mathbb{R}$, the following three properties hold:

a) $|[\{a_n\}] \geq [\text{“0”}]$, with equality if and only if $[\{a_n\}] = [\text{“0”}]$;

b) $\big|[\{a_n\}] \cdot [\{b_n\}]\big| = |[\{a_n\}]| \cdot |[\{b_n\}]|$;

c) $\big|[\{a_n\}] + [\{b_n\}]\big| \leq \big|[\{a_n\}]\big| + \big|[\{b_n\}]\big|$.

In other words, $|\cdot|$ is an absolute value on $\mathbb{R}$.

Proof: Starting with the first property, we have $|a_n| \geq 0$ for all n. Suppose that $\{a_n\} \not\sim$ “0”. Then there must be an $\epsilon > 0$ such that $|a_n| \geq \epsilon$ for infinitely many $n \in \mathbb{N}$. Let $N \in \mathbb{N}$ be such that $|a_n - a_m| < \epsilon/2$ for all $n, m \geq N$. Since there is an $n_0 \geq N$ with $|a_{n_0}| \geq \epsilon$, we have $|a_m| \geq |a_{n_0}| - |a_{n_0} - a_m| > \epsilon - \epsilon/2 = \epsilon/2$ for all $m \geq N$. Thus, $|[\{a_n\}]| = [\{|a_n|\}] > [\text{“0”}]$.

For the second property, using $|a_n b_n| = |a_n| \cdot |b_n|$ for all n, we have

$$\big|[\{a_n\}] \cdot [\{b_n\}]\big| = |[\{a_n b_n\}]| = [\{|a_n b_n|\}] = [\{|a_n|\}] \cdot [\{|b_n|\}] = |[\{a_n\}]| \cdot |[\{b_n\}]|.$$

Now for the third property - the “triangle inequality.” Suppose first that there is some $\epsilon > 0$ and some $N \in \mathbb{N}$ such that $|a_n + b_n| + \epsilon \leq |a_n| + |b_n|$ for all $n \geq N$. Then by the definitions,

$$\big|[\{a_n\}] + [\{b_n\}]\big| = [\{|a_n + b_n|\}] < [\{|a_n| + |b_n|\}] = \big|[\{a_n\}]\big| + \big|[\{b_n\}]\big|.$$

So suppose this is not the case and let $\epsilon > 0$. Then there are infinitely many $n \in \mathbb{N}$ such that $|a_n + b_n| + \epsilon/2 > |a_n| + |b_n|$. Also, there is an $N \in \mathbb{N}$ such that

$$\big||a_n| - |a_m|\big|, \big||b_n| - |b_m|\big|, \big||a_n + b_n| - |a_m + b_m|\big| < \epsilon/6$$

for all $n, m \geq N$. Choose an $n \geq N$ such that $|a_n + b_n| + \epsilon/2 > |a_n| + |b_n|$. Then for all $m \geq N$ we have

$$\begin{aligned} |a_m| + |b_m| - |a_m + b_m| &< |a_n| + \frac{\epsilon}{6} + |b_n| + \frac{\epsilon}{6} - |a_n + b_n| + \frac{\epsilon}{6} \\ &= |a_n| + |b_n| - |a_n + b_n| + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

By the triangle inequality for rational numbers, $|a_m| + |b_m| \geq |a_m + b_m|$. Thus,

$$\big||a_m| + |b_m| - |a_m + b_m|\big| < \epsilon$$

for all $m \geq N$ and $\{|a_n + b_n|\} \sim \{|a_n| + |b_n|\}$. By the definitions, this means that

$$\big|[\{a_n\}] + [\{b_n\}]\big| = \big|[\{a_n\}]\big| + \big|[\{b_n\}]\big|.$$

Lemma: Suppose $[\{a_n\}]$ and $[\{b_n\}]$ are two unequal real numbers. Then there is a rational number c such that $\big|[\{a_n\}] - [\text{“}c\text{”}]\big| < \big|[\{a_n\}] - [\{b_n\}]\big|$.

Proof: Since $\{a_n\} \not\sim \{b_n\}$, $\{a_n - b_n\} \not\sim$ “0”. By Theorem 2, $\big|[\{a_n\}] - [\{b_n\}]\big| = |[\{a_n - b_n\}]| > 0$ so that there is an $\epsilon > 0$ and an $N \in \mathbb{N}$ such that $|a_n - b_n| \geq \epsilon$ for all $n \geq N$. Also, for some $M \in \mathbb{N}$ we have $|a_n - a_m| < \epsilon/2$ for all $n, m \geq M$. Let $c = a_{N+M}$. Then for all $n \geq N + M$, $|a_n - c| < \epsilon/2$ and $|a_n - b_n| \geq \epsilon$. In particular, $|a_n - b_n| - |a_n - c| \geq \epsilon/2$ for all $n \geq N + M$. By the definitions, this means that $\big|[\{a_n\}] - [\text{“}c\text{”}]\big| < \big|[\{a_n\}] - [\{b_n\}]\big|$.

Theorem 3: Every Cauchy sequence of real numbers converges, i.e., the real numbers are a topologically complete field.

[Technically speaking, the ϵ in the definition of Cauchy sequence of real numbers is allowed to be real. However, it suffices to restrict to the case where ϵ is a positive rational number by the lemma above.]

Proof: Let $\{r_n\}$ be a Cauchy sequence of real numbers. If there is an $r \in \mathbb{R}$ and an $N \in \mathbb{N}$ such that $r_n = r$ for all $n \geq N$ we are done, since in this case $\{r_n\} \to r$. So suppose this is not the case. For each $n \in \mathbb{N}$ let $n' \in \mathbb{N}$ be least such that $n' > n$ and $r_n \neq r_{n'}$. For each $n \in \mathbb{N}$ choose (via the lemma above) an $a_n \in \mathbb{Q}$ such that $|r_n - [\text{”}a_n\text{”}]| < |r_n - r_{n'}|$.

Let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $|r_n - r_m| < \epsilon/3$ for all $n, m \geq N$. By the triangle inequality, we have

$$\begin{aligned}|a_n - a_m| = |[``a_n"] - [``a_m"]| &= |[``a_n"] - r_n + r_n - r_m + r_m - [``a_m"]| \\ &\leq |[``a_n"] - r_n| + |r_n - r_m| + |r_m - [``a_m"]| \\ &< |r_n - r_{n'}| + |r_n - r_m| + |r_m - r_{m'}| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

for all $n, m \geq N$. This shows that $\{a_n\}$ is a Cauchy sequence (of rational numbers), i.e., $[\{a_n\}] \in \mathbb{R}$.

Let $\epsilon > 0$ again. Then there are is an $N \in \mathbb{N}$ such that $|r_n - r_m|, |a_n - a_m| < \epsilon/3$ for all $n, m \geq N$. In particular, $|r_N - [``a_N"]| < \epsilon/3$ and also $|[``a_N"] - [\{a_m\}]| < \epsilon/3$. Using the triangle inequality once more,

$$\begin{aligned}|r_n - [\{a_m\}]| &= \left|r_n - r_N + r_N - [``a_N"] + [``a_N"] - [\{a_m\}]\right| \\ &\leq |r_n - r_N| + |r_N - [``a_N"]| + |[``a_N"] - [\{a_m\}]| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

for all $n \geq N$. Thus, $\{r_n\} \to [\{a_m\}] \in \mathbb{R}$.

Lemma: The sum and product of two positive real numbers is positive.

Proof: Suppose $[\{a_n\}]$ and $[\{b_n\}]$ are positive real numbers. Then by definition there are $N_1, N_2 \in \mathbb{N}$ and $\epsilon_1, \epsilon_2 > 0$ such that $a_n \geq \epsilon_1$ for all $n \geq N_1$ and $b_n \geq \epsilon_2$ for all $n \geq N_2$. This implies that $a_n + b_n \geq \epsilon_1 + \epsilon_2 > 0$ and $a_n b_n \geq \epsilon_1 \epsilon_2 > 0$ for all $n \geq \max\{N_1, N_2\}$. In other words, $[\{a_n\}] + [\{b_n\}] = [\{a_n + b_n\}] > 0$ and $[\{a_n\}] \cdot [\{b_n\}] = [\{a_n b_n\}] > 0$.

Theorem 4: The real numbers are totally ordered by $<$.

Proof: Suppose $x < y$. By definition, this means $y - x > 0$. Since $(y + z) - (x + z) = y - x$, we have $y + z > x + z$. Suppose $x < y$ and $y < z$. Then $z - x = (z - y) + (y - x)$ is positive, being the sum of two positive real numbers. Suppose $x < y$ and $z > 0$. Then $z(y - x)$ is positive, so that $zy > zx$.

Suppose $x \not< y$ and $y \not< x$, and write $x = [\{a_n\}], y = [\{b_n\}]$. Let $\epsilon > 0$. There are infinitely many $n \in \mathbb{N}$ such that $a_n - b_n < \epsilon/3$ and infinitely many $n \in \mathbb{N}$ such that $b_n - a_n < \epsilon/3$. Also, there are $N, M \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon/6$ for all $n, m \geq N$ and $|b_n - b_m| < \epsilon/6$ for all

$n, m \geq M$. Choose an $n_0, m_0 \geq \max\{N, M\}$ such that $a_{n_0} - b_{n_0} < \epsilon/3$ and $b_{m_0} - a_{m_0} < \epsilon/3$. Then $b_{n_0} - a_{n_0} < b_{m_0} + \epsilon/6 - a_{m_0} + \epsilon/6 < 2\epsilon/3$. For any $n \geq \max\{N, M\}$ we have

$$\begin{aligned}|a_n - b_n| &= |a_n - a_{n_0} + a_{n_0} - b_{n_0} + b_{n_0} - b_n| \\ &\leq |a_n - a_{n_0}| + |a_{n_0} - b_{n_0}| + |b_{n_0} - b_n| \\ &< \frac{\epsilon}{6} + \frac{2\epsilon}{3} + \frac{\epsilon}{6} \\ &= \epsilon.\end{aligned}$$

Thus $\{a_n - b_n\} \to 0$ and $[\{a_n\}] = [\{b_n\}]$. This shows that either $x < y$ or $y < x$ or $x = y$.

Finally, suppose $x > y$ and $y < x$. Then $[\text{"0"}] = (x - y) + (y - x)$ is positive since it is the sum of two positive numbers. Similarly, if $x = y$ and either $x > y$ or $y > x$, then $[\text{"0"}]$ is positive, which is clearly not the case, so that at most one of $x > y$, $y > x$ and $x = y$ holds.

Theorem 5: Suppose $S \subset \mathbb{R}$, $S \neq \emptyset$, and there is a $B \in \mathbb{R}$ with $x \leq B$ for all $x \in S$. (We say B is an upper bound for S.) Then there is an upper bound $b \in \mathbb{R}$ for S such that for all $c < b$ there is an $x \in S$ with $x > c$. (We say b is a least upper bound for S.)

Proof: Chose an $x_0 \in S$ and let $m_0 \in \mathbb{N}$ be largest such that $B - m_0$ is an upper bound for S. Then m_0 exists since $B - 0$ is an upper bound for S and $B - n$ is not an upper bound for S whenever $n > B - x_0$.

We claim that there is a sequence $B_1, B_2, \ldots$ of upper bounds for S such that $B_n - 2^{-n}$ is not an upper bound for S and $0 \leq B_{n-1} - B_n \leq 2^{-n}$ for all $n \in \mathbb{N}$. We prove this claim by induction.

Let $m_1 \in \mathbb{N}$ be largest such that $B_0 - m_1 2^{-1}$ is an upper bound for S. Then m_1 is either 0 or 1 since $B_0 - 0$ is an upper bound for S and $B_0 - 1 = B_0 - 2 \cdot 2^{-1}$ is not. Let $B_1 = B_0 - m_1 2^{-1}$. Then B_1 is an upper bound for S and $0 \leq m_1 2^{-1} = B_0 - B_1 \leq 2^{-1}$. Also, $B_1 - 2^{-1} = B_0 - (m_1 + 1)2^{-1}$ in not an upper bound for S by the definition of m_1.

Now suppose $B_1, \ldots, B_n$ have been chosen which satisfy the requirements above. Let m_{n+1} be the largest integer such that $B_n - m_{n+1} 2^{-n-1}$ is an upper bound for S. By the induction hypothesis, B_n is an upper bound for S and $B_n - 2^{-n}$ is not, so that m_{n+1} is either 0 or 1. Let $B_{n+1} = B_n - m_{n+1} 2^{-n-1}$. Then B_{n+1} is an upper bound for S and $0 \leq m_{n+1} 2^{-n-1} = B_n - B_{n+1} \leq 2^{-n-1}$. Finally, $B_{n+1} - 2^{-n-1} = B_n - (m_{n+1} + 1)2^{-n-1}$ is not an upper bound for S by the definition of m_{n+1}.

We claim that this is a Cauchy sequence. Indeed, let $\epsilon > 0$ and suppose $n, m \in \mathbb{N}$ with $n > m$.

Then

$$\begin{aligned} B_n - B_m &= (B_n - B_{n-1}) + \cdots + (B_{m+1} - B_m) \\ &\leq 2^{-n} + \cdots + 2^{-m-1} \\ &= 2^{-m-1}(1 + \cdots + 2^{m+1-n}) \\ &= 2^{-m-1}\frac{1 - 2^{m-n}}{1 - 2^{-1}} \\ &= 2^{-m}(1 - 2^{m-n}) \\ &< 2^{-m}. \end{aligned}$$

Thus, if $n, m \geq N$ where $2^{-N} \leq \epsilon$, then $|B_n - B_m| < \epsilon$.

Let $b \in \mathbb{R}$ be the limit of this Cauchy sequence. Let $x \in S$. Suppose $x > b$ and let $\epsilon = x - b$. For some $n \in \mathbb{N}$, $|B_n - b| < \epsilon$. But this implies that $x - b > |B_n - b|$, so that $x > B_n$. This contradicts the fact that all B_n's are upper bounds for S. Thus, b is an upper bound for S.

Finally, suppose $c < b$ is an upper bound for S. Then $b - c \geq 2^{-n_0}$ for some $n_0 \in \mathbb{N}$, which implies that $b - 2^{-n_0}$ is an upper bound for S. Choose $N \in \mathbb{N}$ such that $|B_n - b| < 2^{-n_0-1}$ for all $n \geq N$. We then have $B_n - 2^{-n_0-1} > b - 2^{-n_0}$ for all $n \geq N$. But $B_n - 2^{-n}$ is not an upper bound for any n, so that $B_n - 2^{-n_0-1} \leq B_n - 2^{-n}$ is not an upper bound for any $n > n_0$. This contradition shows that $c \geq b$, so that b is a least upper bound for S.