Definition: A Dirichlet series is a series of the form

\[ D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \]

where \( a_1, a_2, \ldots \) is a sequence of complex numbers.

The quintessential example here is the Riemann zeta function (the case where \( a_n = 1 \) always):

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \]

As with the zeta function, Dirichlet series often have a nice Euler product expansion.

Proposition/Exercise 7: Suppose \( f \) is a multiplicative function and \( \sum \frac{|f(n)|}{n^\sigma} \) converges. Then for all \( s = \sigma + it \) we have

\[ D(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \text{ prime}} \left( 1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots \right). \]

Moreover, if \( f \) is totally multiplicative we have

\[ D(s) = \prod_{p \text{ prime}} \left( 1 - f(p)p^{-s} \right)^{-1}. \]

Using the Proposition we can determine some Dirichlet series.

Example 1: Set \( f(n) = \mu(n) \). Then \( f(p) = -1 \) and \( f(p^k) = 0 \) for all primes \( p \) and \( k > 1 \). We thus have

\[ \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_{p \text{ prime}} \left( 1 - p^{-s} \right) = \frac{1}{\zeta(s)}. \]

This is valid if \( \Re(s) > 1 \).

Example 2: Let \( \lambda \) be the totally multiplicative function given by \( \lambda(p) = -1 \) for all primes \( p \) (this completely determines \( \lambda \)). Then

\[ \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_{p \text{ prime}} \left( 1 - p^{-s} + p^{-2s} - p^{-3s} + \cdots \right) \]
\[ = \prod_{p \text{ prime}} (1 + p^{-s})^{-1} \]
\[ = \prod_{p \text{ prime}} \frac{1 - p^{-s}}{1 - p^{-2s}} \]
\[ = \frac{\zeta(2s)}{\zeta(s)}. \]

This is valid if \( \Re(s) > 1 \).
We will now look at some analytic properties of Dirichlet series.

**Theorem 1:** Suppose the Dirichlet series $D(s)$ converges for some $s_0 = \sigma_0 + it_0$ and $C > 0$. Then $D(s)$ converges uniformly on the set

\[ \{ s = \sigma + it : \sigma \geq \sigma_0, |t - t_0| \leq C(\sigma - \sigma_0) \} . \]

Proof: We will use the technique of Abel summation in our proof. This technique will be used extensively in the remainder of these notes as well.

For a positive integer $N$ set $R(N)$ to be the “tail” of the Dirichlet series $D(s_0)$:

\[ R(N) = \sum_{n>N} \frac{a_n}{n^{s_0}} . \]

Then $R(N-1) - R(N)$ is the $N^{th}$ term of the series, $a_N/N^{s_0}$, so that

\[ N^{s_0} (R(N-1) - R(N)) = a_N . \]

Via (1), we get

\[ N \sum_{n=M+1}^{N} \frac{a_n}{n^{s_0}} = \sum_{n=M+1}^{N} (R(n-1) - R(n)) n^{s_0-s} = R(M) M^{s_0-s} - R(N) N^{s_0-s} - \sum_{n=M+1}^{N} R(n-1)((n-1)^{s_0-s} - n^{s_0-s}) , \]

for all positive integers $M$ with $M + 1 \leq N$. We note that

\[ (n-1)^{s_0-s} - n^{s_0-s} = -(s_0-s) \int_{n-1}^{n} u^{s_0-s-1} du . \]

Defining $R(u) = R([u])$, where $[\cdot]$ denotes the greatest integer function, as usual, we get

\[ \sum_{n=M+1}^{N} \frac{a_n}{n^{s_0}} = R(M) M^{s_0-s} - R(N) N^{s_0-s} + (s_0-s) \int_{M}^{N} R(u) u^{s_0-s-1} du . \]

Now by the hypothesis that $D(s_0)$ converges, we have $R(N) \to 0$ as $N \to \infty$. In particular, given an $\epsilon > 0$, there is a positive $M(\epsilon)$ (depending on $\epsilon$) such that $|R(u)| < \epsilon$ for all $u \geq M(\epsilon)$. Suppose $\sigma > \sigma_0$ and $M \geq M(\epsilon)$. Then we have

\[ \left| \sum_{n=M+1}^{N} \frac{a_n}{n^{s_0}} \right| < \epsilon (M^{\sigma_0-\sigma} + N^{\sigma_0-\sigma}) + \epsilon |s_0-s| \int_{M}^{N} u^{\sigma_0-\sigma-1} du . \]

Letting $N \to \infty$ and integrating (assuming $\sigma > \sigma_0$) yields

\[ \left| \sum_{n=M+1}^{\infty} \frac{a_n}{n^{s_0}} \right| < \epsilon M^{\sigma_0-\sigma} \left( 1 + \frac{|s_0-s|}{\sigma - \sigma_0} \right) . \]
Further, for all \( s \) in the set given in the statement of the theorem we have

\[
|s - s_0| \leq \sigma - \sigma_0 + |t - t_0| \leq (C + 1)(\sigma - \sigma_0).
\]

Therefore, for any \( \epsilon > 0 \) there is a positive integer \( N(\epsilon) \) (depending only on \( \epsilon \) and \( C \)) such that

\[
\left| \sum_{n>N} \frac{a_n}{n^s} \right| < \epsilon
\]

for all \( N \geq N(\epsilon) \) and all \( s \) in the set in question.

**Corollary 1:** If \( D(s) \) is a Dirichlet series, then there is a \( \sigma_c \) (possibly \( \pm \infty \)) such that \( D(s) \) converges for all \( s = \sigma + it \) with \( \sigma > \sigma_c \) and for no \( s \) with \( \sigma < \sigma_c \). Further, if \( \sigma_0 > \sigma_c \), then there is a neighborhood of \( s_0 = \sigma_0 + it_0 \) in which \( D(s) \) converges uniformly.

**Corollary 2:** If \( D(s) \) is a Dirichlet series with \( \sigma_c < \infty \), then \( D(s) \) is analytic for all \( s = \sigma + it \) with \( \sigma > \sigma_c \), and

\[
D'(s) = -\sum_{n=1}^{\infty} \frac{a_n \log n}{n^s}
\]

uniformly in the half-plane given by \( \sigma > \sigma_c \).

**Theorem 2:** Let \( D(s) \) be a Dirichlet series with \( \sigma_c \) as above. Write \( A(x) = \sum_{n \leq x} a_n \). If \( \sigma_c < 0 \), then \( A(x) \) is a bounded function (of \( x \)) and

\[
D(s) = s \int_1^{\infty} A(x)x^{-(s+1)} \, dx
\]

for all \( s = \sigma + it \) with \( \sigma > 0 \). If \( \sigma_c \geq 0 \), then

\[
\limsup_{x \to \infty} \left( \frac{\log |A(x)|}{\log x} \right) = \sigma_c
\]

and

\[
D(s) = s \int_1^{\infty} A(x)x^{-(s+1)} \, dx
\]

for all \( s = \sigma + it \) with \( \sigma > \sigma_c \).

**Proof:** In place of (1) above we use

\[
(1') \quad N \sum_{n=1}^{N} \frac{a_n}{n^s} = A(N)N^{-s} + s \int_1^{N} A(x)x^{-(s+1)} \, dx.
\]

Arguing exactly as above (with \( s_0 = 0 \)), we get

\[
(2') \quad \sum_{n=1}^{N} \frac{a_n}{n^s} = A(N)N^{-s} + s \int_1^{N} A(x)x^{-(s+1)} \, dx.
\]
Suppose
\[ \theta > \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x}. \]
Then letting \( N \to \infty \) in (2') yields
\[ D(s) = s \int_0^\infty A(x)x^{-(s+1)} \, ds \]
whenever \( \Re(s) = \sigma > \theta \).

Suppose that \( \sigma_c < 0 \). Then by Corollary 1 above \( \sum a_n \) exists, so that
\[ \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x} = 0 \]
and
\[ D(s) = s \int_0^\infty A(x)x^{-(s+1)} \, ds \]
whenever \( \Re(s) = \sigma > 0 \).

Now suppose that \( \sigma_c \geq 0 \). By Corollary 1 above \( D(s) \) diverges whenever \( \Re(s) = \sigma < \sigma_c \), so we must have
\[ \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x} \geq \sigma_c. \]
Choose a \( \sigma_0 > \sigma_c \). By (2) (with \( M = s = 0 \)) we have
\[ A(N) = -R(N)N^{\sigma_0} + \sigma_0 \int_0^N R(u)u^{\sigma_0-1} \, du, \]
where
\[ R(u) = \sum_{n>u} \frac{a_n}{n^{\sigma_0}}. \]
By the hypothesis that \( \sigma_0 > \sigma_c \), \( R(u) \) is bounded as a function of \( u \). This shows that \( |A(N)| \ll N^{\sigma_0} \) (where the implicit constant doesn’t depend on \( N \)). Whence
\[ \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x} \leq \sigma_0. \]
Thus
\[ \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x} \leq \sigma_c, \]
concluding the proof.

**Definition:** Given a Dirichlet series, the quantity \( \sigma_c \) above is called the **abscissa of convergence**. The **abscissa of absolute convergence**, \( \sigma_a \), is defined to be
\[ \sigma_a = \inf \{ \sigma : D(s) \text{ converges for all } s \in \mathbb{C} \text{ of the form } s = \sigma + it \}. \]
Though a Dirichlet series may converge at a given value of $s$, that doesn’t imply it converges absolutely. For example, consider the Dirichlet series
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s},
\]
which converges (as we’ve seen) for all $s$ with $\Re(s) > 0$, but only converges absolutely when $\Re(s) > 1$.

**Lemma:** For any Dirichlet series, $\sigma_a \leq \sigma_c + 1$.

Proof: Let $\epsilon > 0$, so that $\sum a_n n^{-\sigma_c - \epsilon}$ converges. This implies that $|a_n| \ll n^{\sigma_c - \epsilon}$, since the summands must go to zero. Thus $\sum |a_n| n^{-\sigma_c - 1+2\epsilon}$ is convergent, implying that $\sigma_a \leq \sigma_c + 2\epsilon$.

**Theorem 3:** Suppose $D(s)$ is a Dirichlet series with $\sigma_c < \infty$. Let $0 < \epsilon < \delta < 1$. Then
\[
|D(s)| \ll (1 + |t|)^{1-\delta+\epsilon}
\]
for all $s = \sigma + it$ with $\sigma \geq \sigma_c + \delta$, where the implicit constant depends on $\epsilon$ and $\delta$.

Proof: Set $s_0 = \sigma_c + \epsilon$ in (2). Assuming that $s = \sigma + it$ with $\sigma \geq \sigma_c + \delta > \sigma_c + \epsilon$, we may let $N \to \infty$ and get
\[
D(s) - \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=M+1}^{\infty} \frac{a_n}{n^s} = R(M)M^{\sigma_c+\epsilon-s} + (\sigma_c + \epsilon - s) \int_{M}^{\infty} R(u)u^{\sigma_c+\epsilon-s-1} du.
\]
Now $D(\sigma_c + \epsilon)$ is convergent, so we know that $|a_n| \ll n^{\sigma_c + \epsilon}$ and $|R(u)| \ll 1$. Thus
\[
|D(s)| \ll \sum_{n=1}^{M} n^{\sigma_c + \epsilon-s} + M^{\epsilon-\delta} + (\delta - \epsilon + |t|) \int_{M}^{\infty} u^{\epsilon-\delta-1} du
\ll M^{1+\epsilon-\delta} + (1 + |t|)M^{\epsilon-\delta}.
\]
Setting $M = [1 + |t|]$ completes the proof.

**Theorem 4:** Let $\sigma_0 \in \mathbb{R}$ and suppose that $\sum a_n n^{-s} = \sum b_n n^{-s}$ for all $s = \sigma + it$ with $\sigma > \sigma_0$. Then $a_n = b_n$ for all $n$.

Proof: Set $c_n = a_n - b_n$, so that $D(s) = \sum c_n n^{-s} = 0$ for all $s = \sigma + it$ with $\sigma > \sigma_0$. In particular, we have $\sigma_c \leq \sigma_0$ for this Dirichlet series. By the Lemma, $\sum |c_n| n^{-s}$ converges whenever $\sigma > \sigma_0 + 1$.

Suppose $c_N \neq 0$ but $c_n = 0$ for all $n < N$. Then as long as $\sigma > \sigma_0$ we have $D(s) = 0$. Very similar to (1), we get
\[
c_N = - \sum_{n>N} c_n (N/n)^{\sigma}.
\]
Set $\delta = |c_N| > 0$, choose a $\sigma_1 > \sigma_0 + 1$ and write
\[
B = \sum_{n>N} |c_n| (N/n)^{\sigma_1}.
\]
Since
\[
\lim_{\sigma \to \infty} \left( \frac{N}{N + 1} \right)^{\sigma - \sigma_1} = 0,
\]

5
there is a \( \sigma_2 > \sigma_1 \) such that \( \left( \frac{N}{N+1} \right)^{\sigma_2 - \sigma_1} < \delta/B \) for all \( \sigma > \sigma_2 \). Now we have

\[
\delta = |c_N| \leq \sum_{n>N} |c_n|(N/n)^{\sigma_2+1}
= \sum_{n>N} |c_n|(N/n)^{\sigma_2+1-\sigma_1}(N/n)^{\sigma_1}
\leq \sum_{n>N} |c_n|(N/n)^{\sigma_1} \left( \frac{N}{N+1} \right)^{\sigma_2+1-\sigma_1}
< B(\delta/B)
= \delta.
\]

This contradiction shows that there is no such \( N \), i.e., \( c_n = 0 \) always.

**Theorem (Landau):** Suppose the Dirichlet series \( \sum a_n n^{-s} \) has a finite abscissa of convergence further that there is some \( n_0 \) such that \( a_n \geq 0 \) for all \( n \geq n_0 \). Then \( s = \sigma_c \) is a singular point for \( D(s) \).

Proof: Without loss of generality (simply subtract off \( \sum_{n<n_0} a_n n^{-s} \)) we may assume that \( n_0 = 1 \). Moreover, after possibly replacing \( a_n \) with \( a_n n^{-\sigma_c} \), we may assume that the abscissa of convergence is 0.

Under our assumptions, \( D(s) \) is analytic on the half-plane \( \Re(s) > 0 \) by Corollary 2. Suppose that \( D(s) \) is also analytic at \( s = 0 \). Then it is analytic in a region of the form

\[
R = \{ s = \sigma + it : \sigma > 0 \text{ or } |s| < \delta \}
\]
for some \( \delta > 0 \).

Now write \( D(s) \) as a power series centered at 1:

\[
D(s) = \sum_{k=0}^{\infty} c_k (s-1)^k.
\]

As usual,

\[
c_k = \frac{D^{(k)}(1)}{k!}
\]
for all \( k \). By Corollary 2 (since \( 1 > \sigma_c = 0 \)) we have

\[
D^{(k)}(1) = \sum_{n=1}^{\infty} a_n (-\log n)^k n^{-1},
\]
so that

\[
D(s) = \sum_{k=0}^{\infty} \frac{(s-1)^k}{k!} \sum_{n=1}^{\infty} \frac{a_n (-\log n)^k}{n}
= \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \sum_{n=1}^{\infty} \frac{a_n (\log n)^k}{n}.
\]

Note that all summands here are real and non-negative as long as \( s \) is real and no greater than 1. In particular, we have absolute convergence for any negative real \( s \) greater than \(-\delta \). But in this case we may
rearrange the series to get

\[
D(s) = \sum_{k=0}^{\infty} \frac{(1 - s)^k}{k!} \sum_{n=1}^{\infty} \frac{a_n (\log n)^k}{n}
\]

\[
= \sum_{n=1}^{\infty} \frac{a_n}{n} \sum_{k=0}^{\infty} \frac{(1 - s)^k (\log n)^k}{k!}
\]

\[
= \sum_{n=1}^{\infty} \frac{a_n}{n} \exp \left( (1 - s) \log n \right)
\]

\[
= \sum_{n=1}^{\infty} a_n n^{\sigma}.
\]

Again, this converges for all real \( s \) with \(-\delta < s \leq 0\). This contradicts the assumption that \( \sigma_c = 0 \), which implies that the Dirichlet series \( D(s) \) does not converge for any negative real \( s \) by Corollary 1.