17. Let $G$ be a group and suppose $H_i \subseteq G$ is a subgroup of $G$ for all $i \in I$. Let $H = \bigcap_{i \in I} H_i$. Since the identity element $e \in H_i$ for all $i$, $e \in H$ and $H$ is not empty. Now suppose $a, b \in H$. Then $a, b \in H_i$ for all $i$ and so $ab^{-1} \in H_i$ for all $i$ since each $H_i$ is a subgroup. Thus $ab^{-1} \in H$ and $H$ is a subgroup by Corollary 3.2.3.

21. b) This is essentially just restating the definition set-theoretically:

$$Z(G) = \{x \in G : xa = ax \text{ for all } a \in G\} = \bigcap_{a \in G} \{x \in G : xa = ax\} = \bigcap_{a \in G} C(a).$$

a) Follows from b), #17 and #19 (which we did in class).

24. a) Let $a \in G$. Then by #19 from section 3.1, $(a^{-1})^n = a^{-n} = (a^n)^{-1}$ for all integers $n$. Since $e^{-1} = e$, we see that $(a^{-1})^n = e$ if and only if $a^n = e$. This shows that the order of $a^{-1}$ is equal to the order of $a$ (even if it is infinite).

b) Let $a, b \in G$ and suppose $m$ is a positive integer. Then by associativity

$$(ab)^m = \underbrace{(ab)(ab) \cdots (ab)}_{m \text{ times}} = a \underbrace{(ba)(ba) \cdots (ba)}_{m-1 \text{ times}} b = a(ba)^{m-1}b.$$ 

Now suppose $(ab)^n = e$. Then by what we just showed,

$$aeb = ab = a(aba^{-1}) = (ab)^{n+1} = a(ba)^n b,$$

so that $(ba)^n = e$ by right and left cancellation. This shows that the order of $ba$ is no greater than the order of $ab$. Of course, this argument is entirely symmetric (just switch the roles of $a$ and $b$), so that the order of $ab$ is no greater than the order of $ba$. Thus, they have the same order (even if it is infinite).

c) Using b) and associativity, for any $a, b \in G$ we have

$$o(aba^{-1}) = o(a(ba^{-1})) = o((ba^{-1})a) = o(b(a^{-1}a)) = o(b).$$

8. Suppose $G_1$ and $G_2$ are groups with identity elements $e_1$ and $e_2$, and subgroups $H_1$ and $H_2$, respectively. Then $e_1 \in H_1$ and $e_2 \in H_2$, so that $(e_1, e_2) \in H_1 \times H_2$. In particular $H_1 \times H_2$ is
not empty. Let \((g_1, g_2)\) and \((h_1, h_2)\) be elements of \(H_1 \times H_2\). Then \(g_1, h_1 \in H_1\) and \(g_2, h_2 \in H_2\), so that \(g_1 h_1^{-1} \in H_1\) and \(g_2 h_2^{-1} \in H_2\). Thus,

\[(g_1, g_2) \cdot (h_1, h_2)^{-1} = (g_1, g_2) \cdot (h_1^{-1}, h_2^{-1}) = (g_1 h_1^{-1}, g_2 h_2^{-1}) \in H_1 \times H_2\]

and \(H_1 \times H_2\) is a subgroup of \(G_1 \times G_2\) by Corollary 3.2.3.