Math 230 Section 1 Quiz #7
Solutions

1. Find a polynomial that approximates \( \sin x \) to within \( 10^{-10} \) for all \( x \) in the interval \([-\pi/2, \pi/2]\).

The \( n^{th} \) degree Taylor polynomial is “off” by \( \frac{f^{(n+1)}(c)x^{n+1}}{(n+1)!} \), where \( c \) is some number between the center (0 in this case) and \( x \). Since \( f(x) = \sin x \) here, \( f^{(n+1)}(c) \) is \( \pm \sin c \) or \( \pm \cos c \). In any case, \( |f^{(n+1)}(c)| \leq 1 \). Also, for our range of \( x \) we have \( |x|^{n+1} \leq (\pi/2)^{n+1} \). So it suffices to find an \( n \) with \( \frac{(\pi/2)^{n+1}}{(n+1)!} < 10^{-10} \). Even without a calculator, you could use \( \pi/2 < 2 \) and easily determine that \( 2^{20}/20! < 10^{-10} \) (it isn’t difficult to see that \( 10! > 2^{20} \), and certainly \( 11 \cdot 12 \cdot \ldots \cdot 20 > 10^{10} \)). So the degree 19 Taylor polynomial will certainly do:

\[
x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{17}}{17!} - \frac{x^{19}}{19!}.
\]

Of course, with a calculator one can find an adequate \( n \) that is smaller.

2. What does it mean to say that the infinite series \( \sum a_n \) converges?

The infinite series, or “improper sum” \( \sum a_n \), is the limit of the sequence of partial sums \( \{s_n\} \); to say that the series converges means that the sequence of partial sums converges.

3. Write out the first few terms of the series \( \sum_{n=0}^{\infty} \left( \frac{5}{2^n} + \frac{1}{3^n} \right) \) to see how it starts, then find the sum of the series.

The first four terms are \( \frac{5}{1} + \frac{1}{1}, \frac{5}{2} + \frac{1}{3}, \frac{5}{4} + \frac{1}{9} \) and \( \frac{5}{8} + \frac{1}{27} \). This series is the sum of the geometric series with ratio 1/2 starting with 5 and the geometric series with ratio 1/3 starting with 1. The partial sums for the first geometric series here are

\[
5 \cdot \frac{1 - (1/2)^n}{1 - (1/2)} \to 5 \cdot \frac{1}{1 - (1/2)} \quad \text{as } n \to \infty
\]

and the partial sums for the second geometric series are

\[
\frac{1 - (1/3)^n}{1 - (1/3)} \to \frac{1}{1 - (1/3)} \quad \text{as } n \to \infty.
\]

So the sum of the series is

\[
5 \cdot \frac{1}{1 - (1/2)} + \frac{1}{1 - (1/3)} = 10 + 3/2.
\]