Our goal is to prove the following.

**Theorem** (Bertrand’s Postulate): For every positive integer \( n \), there is a prime \( p \) satisfying \( n < p \leq 2n \).

We remark that Bertrand’s Postulate is true by inspection for \( n = 1, 2, 3 \) and 4, so from now on we may assume that \( n \geq 5 \).

To prove Bertrand’s Postulate we will derive an upper bound on the integers \( n \) for which the desired property does not hold. Then it will simply be a matter of verifying Bertrand’s Postulate up to that point.

This approach will revolve around the binomial coefficient \( \binom{2n}{n} \).

**Lemma 1:** If there is no prime \( p \) with \( n < p \leq 2n \), then any prime factor of \( \binom{2n}{n} \) is no greater than \( n \).

**Proof:** We easily have
\[
\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{(2n)(2n-1)\cdots(n+1)}{n!}.
\]
Clearly any prime factor of this quantity is no greater than the largest factor in the numerator, \( 2n \).

**Lemma 2:** For any positive integer \( m \) and any prime \( p \),
\[
\text{ord}_p(m!) = \sum_{r \geq 1} \left\lfloor \frac{m}{p^r} \right\rfloor,
\]
where \( \lfloor \cdot \rfloor \) denotes the greatest integer function.

**Proof:** Fix an exponent \( r \) for the moment. The positive integers no larger than \( m \) that are multiples of \( p^r \) are
\[
p^r, 2p^r, \ldots, \left\lfloor \frac{m}{p^r} \right\rfloor p^r
\]
and those that are multiples of \( p^{r+1} \) are
\[
p^{r+1}, 2p^{r+1}, \ldots, \left\lfloor \frac{m}{p^{r+1}} \right\rfloor p^{r+1}.
\]
Thus there are precisely \( \lfloor m/p^r \rfloor - \lfloor m/p^{r+1} \rfloor \) positive integers \( n \leq m \) with \( \text{ord}_p(n) = r \). We now have
\[
\text{ord}_p(m!) = \sum_{n=1}^{m} \text{ord}_p(n)
\]
\[
= \sum_{r \geq 1} \sum_{1 \leq n \leq m, \text{ord}_p(n) = r} r
\]
\[
= \sum_{r \geq 1} r \left( \lfloor m/p^r \rfloor - \lfloor m/p^{r+1} \rfloor \right)
\]
\[
= \sum_{r \geq 1} r \lfloor m/p^r \rfloor - \sum_{r \geq 1} r \lfloor m/p^{r+1} \rfloor
\]
\[
= \sum_{r \geq 1} r \lfloor m/p^r \rfloor - \sum_{r \geq 1} (r-1) \lfloor m/p^r \rfloor
\]
\[
= \sum_{r \geq 1} \left\lfloor \frac{m}{p^r} \right\rfloor.
\]
**Lemma 3:** Suppose \( n \geq 5 \) and \( p \) is a prime divisor of \( \binom{2n}{n} \). If \( p \leq n \), then \( p \leq 2n/3 \).

**Proof:** Suppose \( p \) is a prime divisor of \( \binom{2n}{n} \) and \( p \leq n \). If \( p > 2n/3 \), then \( p^2 > 4n^2/9 > 2n \) since \( n \geq 5 \).

Hence by Lemma 2,

\[
\text{ord}_p((2n)!) = \sum_{r \geq 1} \left\lfloor \frac{2n}{p^r} \right\rfloor - \left\lfloor \frac{n}{p} \right\rfloor.
\]

and

\[
\text{ord}_p(n!) = \sum_{r \geq 1} \left\lfloor \frac{n}{p^r} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor.
\]

Now since \( n \geq p > 2n/3 \) we have \( 3/2 > n/p \geq 1 \), so that

\[
\text{ord}_p\left(\binom{2n}{n}\right) = \text{ord}_p\left(\frac{(2n)!}{(n!)^2}\right) = \text{ord}_p((2n)! \cdot 2\text{ord}_p(n!)) = 2 - 2 = 0.
\]

In other words, \( p \) is not a factor of the binomial coefficient \( \binom{2n}{n} \).

We will now consider the arithmetic function

\[
\theta(x) = \sum_{\substack{p \leq x \leq \pi \text{ prime}}} \log p.
\]

**Lemma 4:** Assume \( n \geq 5 \) is such that there are no primes \( p \) satisfying \( n < p \leq 2n \). Then

\[
\log\left(\binom{2n}{n}\right) \leq \theta(2n/3) + \sqrt{2n} \log(2n).
\]

**Proof:** As in the proof of Lemma 3, for all primes \( p \) we have

\[
\text{ord}_p\left(\binom{2n}{n}\right) = \sum_{r \geq 1} \left\lfloor \frac{2n}{p^r} \right\rfloor - 2 \left\lfloor \frac{n}{p^r} \right\rfloor \\
\leq \sum_{r \geq 1} 1 \\
= \left\lfloor \frac{\log(2n)}{\log p} \right\rfloor.
\]

Note that this order is necessarily at most 1 if \( p > \sqrt{2n} \). Now by Lemma 1 any prime \( p \) dividing \( \binom{2n}{n} \) must
satisfy \( p \leq n \), and by Lemma 3 such a prime necessarily satisfies \( p \leq 2n/3 \). Therefore

\[
\log \left( \binom{2n}{n} \right) = \sum_{\substack{p \mid \binom{2n}{n} \\ p \text{ prime}}} \operatorname{ord}_p \left( \binom{2n}{n} \right) \log p
\]

\[
= \sum_{\operatorname{ord}_p \left( \binom{2n}{n} \right) = 1} \log p + \sum_{\operatorname{ord}_p \left( \binom{2n}{n} \right) > 1} \operatorname{ord}_p \left( \binom{2n}{n} \right) \log p
\]

\[
\leq \sum_{p \leq 2n/3} \log p + \sum_{\operatorname{ord}_p \left( \binom{2n}{n} \right) > 1} \left\lfloor \frac{\log(2n)}{\log p} \right\rfloor \log p
\]

\[
\leq \theta(2n/3) + \sum_{p \leq \sqrt{2n}} \log(2n)
\]

\[
\leq \theta(2n/3) + \sqrt{2n} \log(2n).
\]

We want to use the inequality in Lemma 4 to deduce an upper bound for \( n \). Towards that end we need a large lower bound for \( \log\left( \binom{2n}{n} \right) \). This will lead to a sharper upper bound for the theta function as well.

**Lemma 5:** For all positive integers \( n \),

\[
\frac{2^{2n}}{2\sqrt{n}} < \binom{2n}{n} < \frac{2^{2n}}{\sqrt{2n} + 1}.
\]

**Proof:** Consider the product

\[
P_n := \prod_{i \leq n} \frac{(2i - 1)}{(2i)}
\]

\[
= \frac{(2n)!}{2^{2n} (n!)^2}
\]

\[
= \binom{2n}{n} \frac{1}{2^{2n}}.
\]

Note that

\[
\frac{(2i - 1)(2i + 1)}{(2i)^2} < 1
\]

for all \( i \geq 1 \). Thus \( 1 > (2n + 1)P_n^2 \). This gives the upper bound in the lemma. On the other hand,

\[
1 - \frac{1}{(2i - 1)^2} < 1
\]

for all \( i \geq 1 \), so that

\[
1 > \prod_{i=2}^{n} \left( 1 - \frac{1}{(2i - 1)^2} \right)
\]

\[
= \prod_{i=2}^{n} \frac{(2i - 1)^2 - 1}{(2i - 1)^2}
\]

\[
= \prod_{i=2}^{n} \frac{(2i - 2)(2i)}{(2i - 1)^2}
\]

\[
= \frac{1}{4nP_n^2}.
\]
yielding the other inequality.

**Lemma 6**: For all positive integers $n$,

$$\theta(n) < 2n \log 2.$$

**Proof**: By Lemma 5

$$\log \left( \frac{\binom{2n}{n}}{\binom{n}{n}} \right) = \log \left( \frac{\binom{2n}{n}}{\binom{n}{n}} \right) - \log 2 \leq 2n \log 2 - \frac{1}{2} \log(2n) - \log 2$$

$$= (2n - 1) \log 2 - \frac{1}{2} \log(2n + 1).$$

Since

$$\frac{\binom{2n}{n}}{\binom{n}{n}} = \frac{(2n)!}{n!} \cdot \frac{n}{(n-1)!} = \frac{(2n-1)!}{n!(n-1)!} = \binom{2n-1}{n-1},$$

we have via Lemma 2

$$\log \left( \frac{\binom{2n}{n}}{\binom{n}{n}} \right) = \log \left( \frac{\binom{2n-1}{n-1}}{\binom{n-1}{n-1}} \right)$$

$$= \sum_{p \text{ prime}} \operatorname{ord}_p((2n-1)! \log p - \sum_{p \text{ prime}} \operatorname{ord}_p((n-1)! \log p - \sum_{p \text{ prime}} \operatorname{ord}_p(n!) \log p$$

$$= \sum_{p \text{ prime}} \log p \sum_{r \geq 1} \left( \frac{(2n-1)}{p^r} - \frac{(n-1)}{p^r} - \frac{n}{p^r} \right)$$

$$\geq \sum_{p \text{ prime}} \log p \sum_{n < p \leq 2n-1} \left( \frac{(2n-1)}{p^r} - \frac{(n-1)}{p^r} - \frac{n}{p^r} \right)$$

$$= \theta(2n-1) - \theta(n),$$

whence

$$\theta(2n-1) - \theta(n) < (2n-1) \log 2 - \frac{1}{2} \log(2n + 1).$$

We now proceed by induction. The lemma is true by inspection for 1 and 2, so suppose that $m > 2$ and the lemma is true for all integers $n < m$. If $m$ is odd, then $m = 2n - 1$ for some integer $n \geq 2$ since $m > 2$. By the above inequality and the induction hypothesis,

$$\theta(m) = \theta(2n-1) < \theta(n) + (2n-1) \log 2 - \frac{1}{2} \log(2n + 1)$$

$$< 2n \log 2 + (2n-1) \log 2 - \frac{1}{2} \log(2n)$$

$$= (4n - 1) \log 2 - \frac{1}{2} \log(2n)$$

$$\leq (4n - 2) \log 2 \quad \text{(since } n \geq 2)$$

$$= 2m \log 2.$$

If $m$ is even, then $m = 2n$ for some integer $n$ with $m > n \geq 2$ and $m$ is not a prime. Thus $\theta(m) = \theta(m-1)$ and by what we have already shown,

$$\theta(m) = \theta(m-1) < 2(m-1) \log 2 < 2m \log 2.$$
**Proposition:** Suppose \( n \geq 5 \) and there is no prime \( p \) with \( n \leq p < 2n \). Then

\[
(2n - 1) \log 2 - \frac{1}{2} \log n < \frac{4n}{3} \log 2 + \sqrt{2n \log(2n)}.
\]

Proof: By Lemmas 4, 5 and 6

\[
(2n - 1) \log 2 - \frac{1}{2} \log n < \log \left( \frac{2n}{n} \right) \leq \theta(2n/3) + \sqrt{2n \log(2n)} < \frac{4n}{3} \log 2 + \sqrt{2n \log(2n)}.
\]

**Corollary:** If

\[
(2n - 1) \log 2 - \frac{1}{2} \log n < \frac{4n}{3} \log 2 + \sqrt{2n \log(2n)},
\]

then \( n < 500 \). In particular, Bertrand’s Postulate is true for all \( n \geq 500 \).

Proof: We will assume the inequality holds with \( n \geq 500 \) and obtain a contradiction. First we rearrange the inequality to get

\[
((2n/3) - 1) \log 2 < (\sqrt{2n} + (1/2)) \log n + \sqrt{2n \log 2}
\]

\[
(2/3)n - 1 < \frac{(1 + 2\sqrt{2n}) \log n}{2 \log 2} + \sqrt{2n}.
\]

Consider the function \( f(x) = \log x/x \). Since \( f'(x) = x^{-2}(1 - \log x) \), \( f(x) \) is decreasing for \( x \geq e \). Set \( c = \log(500)/\sqrt{500} \). Then \( \log x \leq c\sqrt{x} \) for all \( x \geq 500 \). In particular, since \( n \geq 500 \) by hypothesis we have

(with the help of a calculator)

\[
(2/3)n - 1 < \frac{(1 + 2\sqrt{2n}) \log n}{2 \log 2} + \sqrt{2n} < 1.62\sqrt{n} + .5671n,
\]

so that

\[
.099n - 1 - 1.62\sqrt{n} < 0.
\]

Now we consider the function \( g(x) = .099x - 1 - 1.62\sqrt{x} \). Since \( g'(x) = .099 - .81x^{-1/2} \), we can see that \( g \) is increasing for all \( x \geq (.81/.099)^2 \approx 67 \). In particular, \( g(x) \geq g(500) > 12 \) for all \( x \geq 500 \). This finishes our proof.

Proof of Bertrand’s Postulate: The primes 503, 257, 131, 67, 41, 23, 13 and 7 show that Bertrand’s Postulate is true for all \( 5 \leq n \leq 500 \), since any such number \( n \) satisfies \( n < p \leq 2n \) for at least one of these primes \( p \). Combining this with the Corollary to the Proposition and our initial observation about \( n \leq 4 \) completes the proof.