Notes on Periodic Continued Fractions

For an irrational $\alpha$ we set $\alpha_0 = \alpha$, $a_0 = \lfloor \alpha_0 \rfloor$ and recursively

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n}, \quad a_n = \lfloor \alpha_n \rfloor, \quad n \geq 0.$$ 

Then the continued fraction expansion for $\alpha$ is given by

$$\alpha = [a_0; a_1, a_2, \ldots] = [a_0 : a_1, \ldots, a_{n-1}, \alpha_n].$$

In particular,

(1) \hspace{1cm} \alpha = \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}}

for all $n \geq 0$, where the $p_n$’s and $q_n$’s are defined as before. Notice that

(2) \hspace{1cm} p_{n-1} q_{n-2} - p_{n-2} q_{n-1} = \det \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} = \pm 1.

For a real number $\alpha$ and a matrix $M \in \text{GL}_2(\mathbb{Z})$, say

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

set

$$M(\alpha) = \frac{\alpha A + B}{\alpha C + D}.$$ 

We say two numbers $\alpha$ and $\beta$ are equivalent and write $\alpha \sim \beta$ if $\alpha = M(\beta)$ for some $M \in \text{GL}_2(\mathbb{Z})$. Together (1) and (2) say that $\alpha \sim \alpha_n$ for any $n \geq 0$.

**Lemma 1:** Suppose $\alpha \sim \beta$ with

$$\alpha = \frac{\beta A + B}{\beta C + D},$$

where $C > D > 0$ and $\beta > 1$. Then $\beta = \alpha_{m+1}$ and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} p_m & p_{m-1} \\ q_m & q_{m-1} \end{pmatrix}$$

for some $m \geq 0$, where $p_n/q_n$ are the convergents in the continued fraction expansion of $\alpha$. 

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Proof: Write the rational number $A/C$ as a finite continued fraction: $A/C = [c_0 : c_1, \ldots, c_m]$. We note that this continued fraction is not unique; if $c_m > 1$ then $A/C = [c_0; \ldots, c_{m-1}, c_m - 1, 1]$ and if $c_m = 1$ then $A/C = [c_0; \ldots, c_{m-2}, c_{m-1} + 1]$. The upshot is that we may assume that $m$ satisfies $r_m s_m - r_{m-1} s_m = AD - BC$, where $r_n/s_n$ are the convergents to this continued fraction. Since $A$ and $C$ are relatively prime and $[c_0; c_1, \ldots, c_m] = A/C = r_m/s_m$, we conclude that $A = r_m$ and $C = s_m$.

Now if we consider the linear Diophantine equation $r_m X - s_m Y = AD - BC$, we see that the solutions $X$ and $Y$ are not unique, but $X$ is unique modulo $s_m = C$. Since $0 < s_m - 1 < s_m$ and $0 < D < C = s_m$ by hypothesis, we see that $s_m - 1 = D$ and whence $r_{m-1} = B$.

We now have 

$$\alpha = \frac{\beta r_m + r_{m-1}}{\beta s_m + s_{m-1}} = [c_0; c_1, \ldots, c_m, \beta],$$

which completes the proof.

Lemma 2: Two real irrational numbers $\alpha$ and $\beta$ are equivalent if and only if $\alpha_n = \beta_m$ for some $n, m \geq 0$.

Proof: One direction is simple: if $\alpha_n = \beta_m$ then $\alpha \sim \alpha_n \sim \beta_m \sim \beta$ and thus $\alpha \sim \beta$.

Now assume $\alpha \sim \beta$ and write 

$$\alpha = \frac{\beta A + B}{\beta C + D}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}_2(\mathbb{Z}).$$

Multiply all of $A, B, C$ and $D$ by $-1$ if necessary so that $\beta C + D > 0$. Write $p_n/q_n$ for the convergents in the continued fraction expansion of $\beta$ and set 

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.$$ 

Then by an exercise we have 

$$\alpha = \frac{\beta_{n+1} A_n + B_n}{\beta_{n+1} C_n + D_n}.$$ 

We have 

$$C_n - D_n = C(p_n - p_{n-1}) + D(q_n - q_{n-1}) = (Cp_n/q_n + D)q_n - (Cp_{n-1}/q_{n-1} - D)q_{n-1}$$ 

and also 

$$\left| \beta - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2}, \quad \left| \beta - \frac{p_{n-1}}{q_{n-1}} \right| \leq \frac{1}{q_{n-1}^2}.$$
Therefore, the difference \((C_n - D_n) - (\beta C + D)(q_n - q_{n-1})\) → 0 as \(n \to \infty\). Since \(\beta C + D > 0\), we see that \(C_n - D_n > 0\) for \(n\) sufficiently large. In addition,

\[
D_n = C_{p_n-1} + D_{q_{n-1}} = (C_{p_n-1}/q_{n-1} + D)q_{n-1}
\]

and \(C_{p_n-1}/q_{n-1} + D \to \beta C + D > 0\) as \(n \to \infty\), so that \(D_n > 0\) for \(n\) sufficiently large. Clearly \(\beta_{n+1} > 1\).

Assuming \(n\) is sufficiently large, we have \(C_n > D_n > 0\) and \(\beta_{n+1} > 1\). Fix such an index \(n\). Then by (3) and Lemma 1 we have \(\beta_{n+1} = \alpha_m\) for some \(m\).

**Definition:** We say an irrational \(\alpha\) is **purely periodic** with period of length \(l\) if its continued fraction expansion is of the form

\[
\alpha = [a_0; a_1, \ldots, a_{l-1}],
\]

where the overline denotes a repeating pattern, similar to the standard notation with decimal expansions. We say \(\alpha\) is **periodic** with period of length \(l\) if \(\alpha_n\) is purely periodic with period of length \(l\) for some \(n \geq 0\).

Notice that if \(\alpha\) is purely periodic with period length \(l\), then so is \(\alpha_n\) for all \(n \geq 0\). Also \(\alpha = [a_0; a_1, \ldots, a_{l-1}, \alpha]\) so that \(\alpha_l = \alpha\). Indeed, \(\alpha_{nl} = \alpha\) for all \(n \geq 0\). We thus see via (1) that

\[
\alpha = \frac{\alpha_{p_{n-1}l-1} + p_{n-2}}{\alpha_{q_{n-1}l-1} + q_{n-2}}, \quad \alpha^2 q_{n-1} + \alpha (q_{n-2} - p_{n-1}) - p_{n-2} = 0
\]

for all \(n \geq 0\). In particular (since \(\alpha\) is irrational) we see that it is a root of an irreducible quadratic polynomial with integer coefficients. Further, if \(\bar{\alpha}\) denotes the “conjugate” root, then we readily see that

\[
\alpha + \bar{\alpha} = \frac{p_{n-1}l-1 - q_{n-2}}{q_{n-1}l-1}
\]

whenever \(nl - 1 \geq 0\). Since \(0 < q_{n-2} < q_{n-1}\) for \(n \geq 2\) and \(p_{n-1}/q_{n-1} \to \alpha\) as \(n \to \infty\), we see that a purely periodic \(\alpha\) is a reduced quadratic irrational number, meaning it is a quadratic irrational greater than 1 where \(-1 < \alpha < 0\).

**Lemma 3:** Suppose \(\alpha = [a_0; a_1, \ldots, a_{l-1}]\) is purely periodic with period of length \(l\). Then for any index \(n\) with \(0 \leq n < l\),

\[
(-1/\bar{\alpha})_n = a_{l-1-n} - \bar{a}_{l-1-n}.
\]
Further, \( [a_{l-1-n} - \alpha_{l-1-n}] = a_{l-1-n} \) and \((-1/\alpha)_l = -1/\alpha\), so that
\[
-1/\alpha = [a_{l-1}; a_{l-2}, \ldots, a_0],
\]

**Proof:** We first consider the case where \( n = 0 \). Since
\[
\alpha = \alpha_l = \frac{1}{\alpha_{l-1} - a_{l-1}},
\]
we see that
\[
(-1/\alpha)_0 = -1/\alpha = a_{l-1} - \alpha_{l-1}.
\]
We now continue on inductively. Assume we’ve shown the lemma for some \( n < l \). Then
\[
(-1/\alpha)_{n+1} = \frac{1}{(-1/\alpha)_n - a_{l-1-n}} = -1/\alpha_{l-1-n}.
\]
We now apply what we proved above to
\[
\alpha_{l-1-n} = [a_{l-1-n}; a_{l-n}, \ldots, a_0, a_1, \ldots, a_{l-2-n}].
\]
If \( n + 1 < l \) then we have
\[
-1/\alpha_{l-1-n} = a_{l-2-n} - \alpha_{l-2-n} = a_{l-1-(n+1)} - \alpha_{l-1-(n+1)}.
\]
If \( n + 1 = l \) then \( \alpha_{l-1-n} = a_0 = \alpha \). Finally, since each \( \alpha_m \) is purely periodic it is reduced. This implies that \( 0 < -\alpha_m < 1 \), so that in particular \([a_{l-1-n} - \alpha_{l-1-n}] = a_{l-1-n} \) whenever \( l - 1 - n \geq 0 \).

**Lemma 4:** If \( \alpha \) is purely periodic, then \( (-\alpha)_n \) is purely periodic for all \( n \geq 1 \).

**Proof:** Since \( \alpha \) is purely periodic, it is reduced. This means that \([-\alpha] = 0\) and thus \((-\alpha)_1 = -1/\alpha\), which is purely periodic by Lemma 3. This shows the case \( n = 1 \). One readily checks that \( \beta_{m+l} = (\beta_m)_l \) for all \( m, l \geq 0 \) and all irrational \( \beta \). The lemma follows.

**Lemma 5:** If \( \alpha \) is a real quadratic irrational number, then it is periodic.

**Proof:** Say \( \alpha \) is a root of the polynomial \( P(X) = aX^2 + bX + c \in \mathbb{Z}[X] \), where \( a > 0 \) and \( b^2 - 4ac > 0 \) is not a perfect square. The equation \( a\alpha^2 + b\alpha + c = 0 \) may be written in matrix terms as
\[
\begin{bmatrix}
\alpha & 1 \\
b/2 & c
\end{bmatrix}
\begin{bmatrix}
\alpha \\
1
\end{bmatrix} = 0.
\]
Now via (1) we see that
\[
(\alpha_n \ 1) \begin{pmatrix} a_n & b_n/2 \\ b_n/2 & c_n \end{pmatrix} \begin{pmatrix} \alpha_n \\ 1 \end{pmatrix} = 0,
\]
where
\[
\begin{pmatrix} a_n & b_n/2 \\ b_n/2 & c_n \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix}.
\]
One readily verifies that
\[
a_n = ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2 = q_{n-1}^2 P(p_{n-1}/q_{n-1}),
\]
\[
c_n = ap_{n-2}^2 + bp_{n-2}q_{n-2} + cq_{n-2}^2 = q_{n-2}^2 P(p_{n-2}/q_{n-2}),
\]
\[
b_n = b^2 - 4ac - 4a_nc_n.
\]
Now the proof of Liouville’s theorem shows that both $|a_n|$ and $|c_n|$ are bounded by fixed functions of $\alpha$, whence $|b_n|$ is as well. Therefore, there are only finitely many possible values for $a_n, b_n$ and $c_n$ here. Since $\alpha_n$ is a root of $a_nX^2 + b_nX + c_n$ for any $n$, we conclude that there are only finitely many possible $\alpha_n$’s. In particular, for some $n \geq 0$ and $l > 0$ we have $\alpha_n = \alpha_{n+l} = (\alpha_n)_l$, so that $\alpha_n$ is purely periodic and $\alpha$ is periodic.

**Theorem:** A real $\alpha$ is purely periodic if and only if it is a reduced quadratic irrational number.

**Proof:** We’ve already seen the easy direction: if $\alpha$ is purely periodic then it is a reduced quadratic irrational. So suppose now that $\alpha$ is a reduced quadratic irrational.

By Lemma 5 $\alpha$ is periodic; we may write
\[
\alpha = [a_0; a_1, \ldots, a_n, \beta]
\]
where $\beta = \alpha_{n+1}$ is purely periodic. Note that while there is a minimal $n$ that works here, any larger index will work as well. Since $\beta = \alpha_{n+1}$, by (1)
\[
\alpha = \frac{\beta p_n + p_{n-1}}{\beta q_n + q_{n-1}},
\]
where the $p_n$’s and $q_n$’s are given by the continued fraction expansion of $\alpha$. By a previous exercise, we get
\[
\beta = \frac{\alpha q_{n-1} - p_{n-1}}{\alpha(-q_n) + p_n}.
\]
Taking conjugates and multiplying through by $-1$ gives
\[
-\beta = \frac{(-1/\alpha)p_{n-1} + q_{n-1}}{(-1/\alpha)p_n + q_n}.
\]
Now since $\alpha$ is reduced, $-1/\alpha > 1$. Since $p_n/q_n \to \alpha > 1$ as $n \to \infty$, $p_n > q_n$ for $n$ sufficiently large and $q_n \geq 1$, too. Assuming our index $n$ is sufficiently large, which we may by our earlier remark above, we conclude via Lemma 1 that $-1/\alpha = (-\beta)_{m+1}$ for some $m \geq 0$. By Lemma 4, $(-\beta)_{m+1}$ is purely periodic since $\beta$ is by hypothesis. Now by Lemma 3,

$$\alpha = \frac{-1}{-1/\alpha}$$

is purely periodic.