1. For a matrix $M \in \text{GL}_2(\mathbb{Z})$, say

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and a number $\alpha$, set

$$M(\alpha) = \frac{\alpha A + B}{\alpha C + D}.$$

As in class, we say two numbers $\alpha$ and $\beta$ are equivalent and write $\alpha \sim \beta$ if $\alpha = M(\beta)$ for some $M \in \text{GL}_2(\mathbb{Z})$. Show that this is an equivalence relation. Specifically, show that $I(\alpha) = \alpha$ for any $\alpha$ (where $I$ is the identity, as usual), if $\alpha = M(\beta)$ then $\beta = M^{-1}(\alpha)$, and if $\beta = M_1(\alpha)$ and $\gamma = M_2(\beta)$ for some $M_1, M_2 \in \text{GL}_2(\mathbb{Z})$, then $\gamma = M_2 M_1(\alpha)$.

2. For any real number $\alpha$ and integer $n$, show that $\alpha + n \sim \alpha$. If $\alpha \neq 0$, show that $\alpha \sim 1/\alpha$.

3. Find the continued fraction expansions for $\sqrt{2}$ and $\frac{\sqrt{2} + 3}{7}$. Conclude that they are equivalent, and find an $M \in \text{GL}_2(\mathbb{Z})$ with

$$\frac{\sqrt{2} + 3}{7} = M(\sqrt{2}).$$

4. Find the continued fraction expansions of $\sqrt{7}$, $\sqrt{10}$, $\sqrt{11}$ and $\sqrt{13}$. What are the periods of each? To which purely periodic number is each of these equivalent?

5. Find the continued fraction expansions of $\sqrt{5}$, $\sqrt{17}$ and $\sqrt{26}$. You’ve already found $\sqrt{2}$ and $\sqrt{10}$. Make a general conjecture about the continued fraction expansion of $\sqrt{n^2 + 1}$. Can you prove your conjecture? What about $\sqrt{n^2 - 1}$?