The Riemann zeta function is defined to be
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]
for \( s > 1 \). The “integral test” from calculus shows that this infinite series converges (absolutely) when \( s > 1 \). One goal is to get a definition for the zeta function that is valid for all values of \( s \) except 1 (it’s not hard to show - and we will - that the zeta function blows up as \( s \) approaches 1). But first, we give some motivation for why a number theorist would care.

**Euler’s Product Formula:** For all \( s > 1 \) we have
\[ \zeta(s) = \prod_{p} (1 - p^{-s})^{-1}, \]
where the (infinite) product is over all prime numbers \( p \).

Thus, the zeta function is intimately connected with the prime numbers. Before we prove this formula for the zeta function, we need a little result.

**Lemma 1:** Suppose \( a_1, \ldots, a_n \) are real numbers satisfying \( 1 \leq a_i \leq 2 \) for all \( i = 1, \ldots, n \). Then for any positive \( b < 1 \) we have
\[ \prod_{i=1}^{n} (a_i + b) < \prod_{i=1}^{n} a_i + 2^{2n}b. \]

**Proof:** Exercise 1.

**Proof of Product Formula:** Since every prime \( p \geq 2 \), we see that the geometric series \( \sum_{n=0}^{\infty} p^{-sn} \) converges whenever \( s > 0 \); in fact
\[ \sum_{n=0}^{\infty} p^{-sn} = \frac{1}{1 - p^{-s}}. \]
Moreover, we even have an exact formula for the partial sums:
\[ \sum_{n=0}^{N} p^{-sn} = \frac{1 - p^{-s(N+1)}}{1 - p^{-s}} = \frac{1}{1 - p^{-s}} - \frac{p^{-s(N+1)}}{1 - p^{-s}}. \]
Thus, assuming that \( s > 1 \) and using \( p \geq 2 \),
\[
0 < (1 - p^{-s})^{-1} - \sum_{n=0}^{N} p^{-sn} < \frac{2^{-(N+1)}}{1 - (1/2)} = 2^{-N}.
\]

Note that we also have
\[
1 \leq \sum_{n=0}^{N} p^{-sn} < 2
\]
as well.

Next, notice what happens when you multiply these finite sums together:
\[
\prod_{p \leq M} \sum_{n=0}^{N} p^{-sn} = \sum_{m} m^{-s},
\]
where the sum on the right is over all integers \( m \) which have a factorization of the form
\[
m = p_{1}^{e_{1}} \cdots p_{r}^{e_{r}}, \quad p_{i} \leq M \text{ and } e_{i} \leq N \text{ for all } i.
\]

In particular,
\[
\zeta(s) > \prod_{p \leq M} \sum_{n=0}^{N} p^{-sn} \geq \zeta(s) - \sum_{n > M} n^{-s}.
\]

Now fix an \( \epsilon > 0 \) and suppose \( s > 1 \). Then there is an \( M_{0} \) such that \( \sum_{n > M_{0}} n^{-s} < \epsilon \) (since the series for \( \zeta(s) \) converges). If \( M \geq M_{0} \), then get an \( N \) such that \( 2^{2\pi(M)-N} < \epsilon \). (That’s the function \( \pi(M) \) in the exponent, not the product of \( M \) and the number \( \pi \).) By Lemma 1 and what we have shown,
\[
\prod_{p \leq M} \sum_{n=0}^{N} p^{-sn} < \prod_{p \leq M} (1 - p^{-s})^{-1} < \prod_{p \leq M} \left( \sum_{n=0}^{N} p^{-sn} + 2^{-N} \right)
\]
\[
\zeta(s) - \sum_{n > M} n^{-s} < \prod_{n \leq M} (1 - p^{-s})^{-1} < \left( \prod_{p \leq M} \sum_{n=0}^{N} p^{-sn} \right) + 2^{2\pi(M)-N}
\]
\[
\zeta(s) - \epsilon < \prod_{p \leq M} (1 - p^{-s})^{-1} < \zeta(s) + \epsilon.
\]

The last two inequalities are valid whenever \( M \geq M_{0} \), and since \( \epsilon > 0 \) was arbitrary, this proves the product formula.

We really would like the zeta function to be defined for more values of \( s \) than just \( s > 1 \). The simplest way to do something like this is the following. Set
\[
f(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.
\]
By the alternating series test from calculus, this series converges whenever \( s > 0 \).

**Lemma 2:** For \( s > 1 \)

\[(1 - 2^{1-s})\zeta(s) = f(s).\]

**Proof:** Exercise 2.

This prompts us to define

\[\zeta(s) = \frac{f(s)}{1 - 2^{1-s}}\]

for all positive \( s \neq 1 \). Of course, we just showed that this is the same as our original definition when \( s > 1 \). We can use this expanded definition of the zeta function to compute a limit which gives us more precise information on how zeta “blows up” at \( s = 1 \) (note that the following *two-sided* limit would not be possible using our original definition).

**Lemma 3:** We have

\[\lim_{s \to 1} (s - 1)\zeta(s) = 1.\]

**Proof:** Exercise 3.

In particular, \( \lim_{s \to 1^+} \zeta(s) = \infty \). This together with the product formula shows that there must be infinitely many primes (otherwise the product would be finite when \( s = 1 \)). This may seem like a rather silly way to prove there are infinitely many primes, but these sorts of ideas can be used to prove more opaque results (e.g., there are infinitely many primes congruent to 3 modulo 7).

While we have managed to expand the domain of the zeta function somewhat, we can do much better. One generally goes about this using complex analysis (meaning analysis involving the complex numbers). That’s out of our league here in Math 480, so we’ll try to just use calculus as much as possible and rely on a few “facts.”

We take a temporary diversion and look at the Gamma function. This may seem to be rather tangential, but in fact the Gamma function and zeta function are closely related.

As with the zeta function, we start with a definition for only some values of \( s \):

\[\Gamma(s) = \int_0^\infty e^{-x}x^{s-1} \, dx \quad \text{for} \ s > 0.\]

When we write an improper integral such as this, it is shorthand for the appropriate one-sided limit(s). Note that the infinite limit of integration is really no problem at all, since \( e^{-x} \) goes to zero quite rapidly. The zero lower limit of integration is what forces us to only allow \( s > 0 \).
The Gamma function has a fairly simple “functional equation” which allows us to define it for all \( s \) except non-positive integers.

**Lemma 4:** For \( s > 0 \)

\[
\Gamma(s + 1) = \int_0^\infty e^{-x}x^s \, dx = s \int_0^\infty e^{-x}x^{s-1} \, dx = s\Gamma(s).
\]

**Proof:** Exercise 4.

Using this gives us a way to define \( \Gamma(s) \) for all \( s \neq 0, -1, -2, \ldots \) For example,

\[
\Gamma(-2.4) = \frac{\Gamma(-1.4)}{-2.4} = \frac{\Gamma(-.4)}{(-1.4)(-2.4)} = \frac{\Gamma(.6)}{(-.4)(-1.4)(-2.4)} = \frac{\int_0^\infty e^{-x}x^{-4} \, dx}{(-.4)(-1.4)(-2.4)}.
\]

It isn’t difficult to determine that \( \Gamma(1) = 1 \) (see below). Thus,

\[
\lim_{s \to 0} s\Gamma(s) = 1,
\]

so that the Gamma function “blows up” like \( 1/s \) for \( s \) near 0.

**Exercise 5:** Use (1) and the functional equation for the Gamma function to evaluate

\[
\lim_{s \to -1} (s + 1)\Gamma(s)
\]

\[
\lim_{s \to -2} (s + 2)\Gamma(s)
\]

\[
\lim_{s \to -3} (s + 3)\Gamma(s).
\]

Find a general formula for

\[
\lim_{s \to -n} (s + n)\Gamma(s)
\]

for all positive integers \( n \).

We explicitly evaluate the Gamma function for a couple values of \( s \). First,

\[
\Gamma(1) = \int_0^\infty e^{-x} \, dx = \lim_{b \to \infty} -e^{-x}\big|_{x=0}^b = \lim_{b \to \infty} 1 - e^{-b} = 1.
\]

Using this, if \( n \) is a positive integer we get

\[
\Gamma(n + 1) = n\Gamma(n) = n(n - 1)\Gamma(n - 1) = \cdots = n!\Gamma(1) = n!.
\]

This also explains why \( 0! \) should be 1, since \( 0! \) should be \( \Gamma(1) \) if we set \( n = 0 \) in our equation above.
We next evaluate $\Gamma(1/2)$. To do this we'll use more advanced calculus (double integrals, to be precise, and changing to polar coordinates). First, if $u^2 = x$, then $2u \, du = dx$ and we have

$$\Gamma(1/2) = \int_0^\infty e^{-x} x^{-1/2} \, dx = \int_0^\infty e^{-u^2} \, du$$

since $u = 0$ when $x = 0$ and $u \to \infty$ when $x \to \infty$. Squaring and passing to polar coordinates gives us

$$\Gamma(1/2)^2 = 4 \int_0^\infty e^{-u^2} \, du \int_0^\infty e^{-v^2} \, dv$$

$$= 4 \int_0^\infty \int_0^\infty e^{-(u^2+v^2)} \, du \, dv$$

$$= 4 \int_0^\infty \int_0^{\pi/2} e^{-r^2} \, r \, d\theta \, dr$$

$$= 2\pi \int_0^\infty e^{-r^2} \, dr$$

$$= \pi \lim_{b \to \infty} -e^{-r^2} \bigg|_0^b$$

$$= \pi.$$

In other words,

$$\Gamma(1/2) = \sqrt{\pi}.$$  

(3) \hspace{1cm} \Gamma(1/2) = \sqrt{\pi}.

**Fact 1:** For all $s$ with neither $s = -n$ nor $s + 1/2 = -n$ for a non-negative integer $n$,

$$\Gamma(s)\Gamma(s + 1/2) = \pi^{1/2} 2^{1-2s} \Gamma(2s).$$

This is called Legendre’s formula.

**Euler’s constant** is the number $\gamma$ defined by

$$\gamma = \lim_{n \to \infty} \left( \sum_{m=1}^n \frac{1}{m} - \log n \right).$$

**Fact 2:** For all $s$

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^\infty \left(1 + \frac{s}{n}\right) e^{-s/n}.$$

This is called Euler’s formula.

This infinite product converges for all $s$, and clearly is equal to zero precisely when $s = 0, -1, -2, \ldots$
**Fact 3**: For all \( s \) not an integer,

\[
\Gamma(s)\Gamma(1-s) = -s\Gamma(s)\Gamma(-s) = \frac{\pi}{\sin(\pi s)}.
\]

One more thing before we get back to the zeta function. We will use a certain type of function called a *theta function* in what follows. Theta functions are used frequently in number theory, and are a subject of study all by themselves. We will use a particular theta function,

\[
\theta(z) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 z} = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 z}.
\]

**Fact 4**: For all positive \( z \)

\[
\sqrt{z}\theta(z) = \theta(1/z).
\]

We now return to the zeta function. So far we’ve seen how to define \( \zeta(s) \) for all positive \( s \neq 1 \), and we’ve seen how \( \zeta(s) \) behaves as \( s \) approaches 1. We’ll prove a functional equation which will allow us to define \( \zeta(s) \) for all \( s \neq 1 \).

Consider the function

\[
F(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).
\]

For now, we just say this defines \( F \) for all positive \( s \neq 1 \). We know, however, what \( F(s) \) should behave like as \( s \) approaches 1; via Lemma 3 and (3)

\[
\lim_{s \to 1} (s-1)F(s) = \pi^{-1/2}\Gamma(1/2) \lim_{s \to 1} (s-1)\zeta(s) = 1.
\]

For the moment, assume \( s > 1 \). Then using our original representation of the zeta function

\[
F(s) = \pi^{-s/2} \sum_{n=1}^{\infty} \frac{1}{n^s} \int_{0}^{\infty} e^{-x} x^{s/2} \frac{dx}{x}.
\]

Take the integral here and make the substitution \( y = ax \), where \( a \) is some positive number. Then \( dy = a \, dx \) and \( \frac{dy}{y} = \frac{a \, dx}{x} = \frac{dx}{x} \). Also, \( y = 0 \) when \( x = 0 \) and \( y \to \infty \) when \( x \to \infty \). We get

\[
\int_{0}^{\infty} e^{-x} x^{s/2} \frac{dx}{x} = \int_{0}^{\infty} e^{-y/a} \frac{dy}{a^{s/2}}.
\]
In particular, if we set \( a = 1/\pi n^2 \), we get
\[
F(s) = \pi^{-s/2} \sum_{n=1}^{\infty} \frac{1}{n^s} \int_{0}^{\infty} e^{-x} x^{s/2} \frac{dx}{x}
\]
\[
= \sum_{n=1}^{\infty} \int_{0}^{\infty} e^{-x} \left( \frac{x}{\pi n^2} \right)^{s/2} \frac{dx}{x}
\]
\[
= \sum_{n=1}^{\infty} \int_{0}^{\infty} e^{-\pi n^2 y} y^{s/2} \frac{dy}{y}
\]
\[
= \int_{0}^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 y} y^{s/2} \frac{dy}{y}.
\]

The last step where we interchange the infinite sum with the integral is justified since everything is absolutely convergent. If we set \( g(z) = \sum_{n=1}^{\infty} e^{-\pi n^2 z} \), then
\[
F(s) = \int_{0}^{\infty} g(y)y^{s/2} \frac{dy}{y}.
\]

As we remarked about the Gamma function, the problem here isn’t the infinite limit of integration so much as the zero lower limit of integration. We’ll get around that by splitting the integral up and doing another change of variables. First
\[
F(s) = \int_{0}^{\infty} g(y)y^{s/2} \frac{dy}{y} = \int_{1}^{\infty} g(y)y^{s/2} \frac{dy}{y} + \int_{0}^{1} g(y)y^{s/2} \frac{dy}{y}.
\]

Now set \( z = 1/y \). Then \( dz = -y^{-2} dy \) and \( dz/z = -dy/y \). Also, \( z = 1 \) when \( y = 1 \) and \( z \to \infty \) as \( y \to 0^+ \). Thus,
\[
F(s) = \int_{1}^{\infty} g(y)y^{s/2} \frac{dy}{y} - \int_{\infty}^{1} g(1/z)z^{-s/2} \frac{dz}{z} = \int_{1}^{\infty} g(y)y^{s/2} \frac{dy}{y} + \int_{1}^{\infty} g(1/z)z^{-s/2} \frac{dz}{z}.
\]

But notice that \( 2g(w) + 1 = \theta(w) \) for any \( w \), so that Fact 4 gives
\[
2g(1/z) + 1 = \theta(1/z) = \sqrt{z}\theta(z) = \sqrt{z} (2g(z) + 1).
\]

Hence
\[
g(1/z) = z^{1/2} g(z) + \frac{z^{1/2} - 1}{2}.
\]

Using this, we have
\[
F(s) = \int_{1}^{\infty} g(y)y^{s/2} \frac{dy}{y} + \int_{1}^{\infty} g(1/z)z^{-s/2} \frac{dz}{z}
\]
\[
= \int_{1}^{\infty} g(y)y^{s/2} \frac{dy}{y} + \int_{1}^{\infty} g(z)z^{(1-s)/2} \frac{dz}{z} + \int_{1}^{\infty} \frac{z^{1/2} - 1}{2} z^{-s/2} \frac{dz}{z}
\]
\[
= \int_{1}^{\infty} g(y) \left( y^{s/2} + y^{(1-s)/2} \right) \frac{dy}{y} + \int_{1}^{\infty} \frac{z^{1/2} - 1}{2} z^{-s/2} \frac{dz}{z}.
\]
This last integral is easily evaluated (remember $s > 1$):

$$\int_{1}^{\infty} \frac{z^{1/2} - 1}{2} z^{-s/2} \frac{dz}{z} = \frac{1}{2} \lim_{b \to \infty} \left[ \frac{1}{z - (s-1)/2} - \frac{1}{z - s/2} \right]_{b}^{\infty}$$

$$= \frac{1}{2} \lim_{b \to \infty} \frac{b^{-(s-1)/2} - b^{-s/2}}{-(s-1)/2 - s/2} - \left( \frac{1}{-(s-1)/2 - s/2} \right)$$

$$= \frac{1}{2} \left( \frac{1}{(s-1)/2} - \frac{1}{s/2} \right)$$

$$= \frac{1}{s - 1} - \frac{1}{s}. $$

This gives us what we sought:

$$F(s) = \frac{1}{s - 1} - \frac{1}{s} + \int_{1}^{\infty} g(y) \left( y^{s/2} + y^{(1-s)/2} \right) \frac{dy}{y}.$$ 

The integral here converges no matter what $s$ is (since it starts at 1 instead of 0). Thus, we have a definition for $F(s)$ whenever $s \neq 0,1$, and we readily see that

$$\lim_{s \to 0} -sF(s) = \lim_{s \to 1} (s - 1)F(s) = 1.$$ 

Since we already had a definition for $\Gamma(s/2)$ for all $s \neq 0, -2, -4, \ldots$ (where it blows up), knowing that

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{s - 1} - \frac{1}{s} + \int_{1}^{\infty} g(y) \left( y^{s/2} + y^{(1-s)/2} \right) \frac{dy}{y}$$

gives us a definition of $\zeta(s)$ for all $s \neq 1$. Notice that to counteract where $\Gamma(s/2)$ blows up, we see that

$$\zeta(s) = 0 \quad \text{for} \quad s = -2, -4, -6, \ldots $$

**Exercise 6:** Determine

$$\lim_{s \to 0} \frac{F(s)}{\Gamma(s/2)}$$

and use this to compute $\zeta(0)$.

Crucially, we notice that

$$F(1 - s) = \frac{1}{(1-s) - 1} - \frac{1}{1 - s} + \int_{1}^{\infty} g(y) \left( y^{(1-s)/2} + y^{(1-(1-s))/2} \right) \frac{dy}{y}$$

$$= \frac{1}{s - 1} - \frac{1}{s} + \int_{1}^{\infty} g(y) \left( y^{(1-s)/2} + y^{s/2} \right) \frac{dy}{y}$$

$$= F(s). $$
In other words, 
\[ \pi^{(s-1)/2}\Gamma((1-s)/2)\zeta(1-s) = \pi^{-s/2}\Gamma(s/2)\zeta(s). \]

This equation can be used to define \( \zeta(s) \) for all \( s \neq 1 \). We can clean this up by using Legendre's formula with \( s/2 \) in place of \( s \) and Euler's formula with \( (s+1)/2 \) in place of \( s \). We get

\[
\zeta(1-s) = \pi^{-s+1/2} \frac{\Gamma(s/2)}{\Gamma(1-(s+1)/2)} \zeta(s)
= \pi^{-s+1/2} \Gamma(s/2) \Gamma((1+s)/2) \sin\left(\pi(1+s)/2\right) \zeta(s)
= \pi^{-s-1/2} \cos(\pi s/2) \pi^{1/2} 2^{1-s} \Gamma(s) \zeta(s)
= \pi^{-s} 2^{1-s} \cos(\pi s/2) \Gamma(s) \zeta(s).
\]

(Here we used the identity from trigonometry \( \sin(\theta + \pi/2) = \cos(\theta) \).) We now have Riemann's functional equation for the zeta function:

\[
\zeta(1-s) = \pi^{-s} 2^{1-s} \cos(\pi s/2) \Gamma(s) \zeta(s).
\]

**Exercise 7:** Use Lemma 3, (1) and the functional equation to compute \( \zeta(0) \).