Notes on the Pell Equation

We’re interested here in solutions $X = x, Y = y \in \mathbb{Z}$ to equations of the form

$$X^2 - dY^2 = \pm 1,$$

where $d$ is a positive integer that is not a square. We can factor here to get $(X - \sqrt{d}Y)(X + \sqrt{d}Y) = \pm 1$. Assuming $x$ and $y$ are both positive, this implies that our solution $(x, y)$ is such that $x/y$ is a good approximation to $\sqrt{d}$. Indeed, we have $|x - \sqrt{dy}| < 1/2$, so that $x \geq \sqrt{dy} - 1/2$ and $x + \sqrt{dy} \geq 2\sqrt{dy} - 1/2$. Since $d \geq 2$, $(2\sqrt{d} - 2)y - 1/2 > 2y/3 - 1/2 > 0$, so that $x + \sqrt{dy} > 2y$ and thus

$$|\sqrt{d} - x/y| < \frac{1}{2y^2}.$$ 

Therefore, it suffices to only consider solutions of the form $x/y = p_n/q_n$, convergents for the continued fraction expansion of $\sqrt{d}$.

It’s easy to see that $\sqrt{d} + [\sqrt{d}]$ is reduced, thus has a purely periodic continued fraction expansion:

$$\sqrt{d} + [\sqrt{d}] = [2a_0; a_1, \ldots, a_{l-1}],$$

where $a_0 = [\sqrt{d}]$. We now see that

$$\sqrt{d} = [a_0; a_1, \ldots, a_{l-1}, 2a_0].$$

For notational convenience temporarily set $\alpha = \sqrt{d}$ and $\beta = \sqrt{d} + [\sqrt{d}] = \alpha + a_0$. We thus have

\begin{equation}
\alpha_n = \beta_n, \quad n \geq 1 \\
\beta_{nl} = \beta, \quad n \geq 0.
\end{equation}

If $p_n/q_n$ denote the convergents in the continued fraction expansion of $\alpha$, then by (1) we have

$$\alpha = \frac{\beta p_{nl-1} + p_{nl-2}}{\beta q_{nl-1} + q_{nl-2}}$$

for all $n \geq 1$. By a previous exercise,

$$\beta = \frac{\alpha q_{nl-2} - p_{nl-2}}{\alpha (-q_{nl-1}) + p_{nl-1}} = \frac{dq_{nl-1}q_{nl-2} - p_{nl-1}p_{nl-2}}{p_{nl-1}^2 - dq_{nl-1}^2} + \sqrt{d} \frac{p_{nl-1}q_{nl-2} - p_{nl-2}q_{nl-1}}{p_{nl-1}^2 - dq_{nl-1}^2}. $$

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We thus see that
\[ p_{nl}^2 - dq_{nl}^2 = p_{nl-1}q_{nl-2} - p_{nl-2}q_{nl-1} = \begin{cases} 1 & \text{if } nl \text{ is even,} \\ -1 & \text{if } nl \text{ is odd.} \end{cases} \]

Note that the second case above is only possible if \( l \) is odd.

More generally, for any \( m \geq 1 \) we have
\[ \beta_m = \frac{\alpha q_{m-2} - p_{m-2}}{\alpha(-q_{m-1}) + p_{m-1}} = \frac{dq_{m-1}q_{m-2} - pm-1p_{m-2}}{p_{m-1}^2 - dq_{m-1}^2} + \sqrt{d} \frac{(-1)^m}{p_{m-1} - dq_{m-1}}. \]

Note that if \( m \) is even then \( p_{m-1}/q_{m-1} > \sqrt{d} \), and if \( m \) is odd then \( p_{m-1}/q_{m-1} < \sqrt{d} \). Therefore
\[ \frac{(-1)^m}{p_{m-1} - dq_{m-1}} > 0. \]

Now if \( p_{m-1}^2 - dq_{m-1}^2 = \pm 1 \) we must have \( p_{m-1}^2 - dq_{m-1}^2 = (-1)^m \) and thus
\[ \beta_m = (-1)^m (dq_{m-1}q_{m-2} - pm-1p_{m-2}) + \sqrt{d}. \]

But since \( \beta \) is purely periodic, so is \( \beta_n \) for any \( n \). This implies that \( \beta_m \) is reduced, whence \( \beta_m = [\sqrt{d}] + \sqrt{d} = \beta \). We have proven the following.

**Theorem:** Suppose \( d \) is a positive integer, not a square. Let \( l \) denote the length of the period in the continued fraction expansion of \( \sqrt{d} \) and let \( p_n/q_n \) denote the convergents in this continued fraction. Then the positive solutions \( x, y \in \mathbb{Z} \) to the equation \( x^2 - dy^2 = 1 \) are \( x = p_{nl-1}, \ y = q_{nl-1} \) where \( n \geq 1 \) is such that \( nl \) is even. The positive solutions \( x, y \in \mathbb{Z} \) to the equation \( x^2 - dy^2 = -1 \) are \( x = p_{nl-1}, \ y = q_{nl-1} \) where \( n \geq 1 \) is such that \( nl \) is odd. In particular, this equation has no solutions in integers unless \( l \) is odd.