REVIEW OUTLINE FOR EXAM 1: Sections 6.1 and 6.2

General note: Many of you have grown to be competent at doing solids of revolution problems. If you are one of those people, just go straight to the problems lists and do a few to refresh your skills. The slicing problems that are NOT solids of revolution may require more practice, since we have done fewer of them. Make sure to do some.

Section 6.1: Volumes by Slicing

Recall our basic *definition* of the volume of a solid with known cross sectional area:

- The **volume** of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of $A(x)$ from a to b. In symbols:
$$V = \int_a^b A(x)\ dx.$$
[Volumes of solids of revolution computed using the washer/disc method are special cases of volumes by slicing: If we slice perpendicular to the axis of revolution, the cross-sections are washers or discs and our areas $A(x)$ are areas of circles or differences between areas of circles.]

An important point to realize is that in a given problem, once we have identified a solid as one of "integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$", our analytical work is done. We just have to evaluate the integral.

Problem. A solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. What is its volume?
Solution. The cross-sections range from $x = -1$ to $x = 1$. The areas $A(x)$ of these cross sections are areas of discs whose diameters vary with x. The length $D(x)$ of the diameter at the cross section at x is the distance between (the y-coordinates) x^2 and $2 - x^2$. That is, $D(x) = |x^2 - (2 - x^2)| = |2x^2 - 2|$. Since $-1 \le x \le 1$, $2x^2 \le 2$, so $D(x) = |2x^2 - 2| = 2 - 2x^2$. Thus, $A(x) = \pi\left(\frac{D(x)}{2}\right)^2 = \pi(1 - x^2)^2$, and $V = \int_{-1}^{1} A(x)\ dx = \int_{-1}^{1} \pi(1 - x^2)^2\ dx$.
[Note: If we had drawn graphs of $y = x^2$ and $y = 2 - x^2$, we would have seen immediately that $2 - x^2 \ge x^2$ on $[-1, 1]$ so that $D(x) = \textit{larger} - \textit{smaller} = (2 - x^2) - x^2 = 2 - 2x^2$. Use graphs when you can!]

Problems you should be able to do: p.321: 1,2,3,5,6,7,8; p.362: 1-6

As noted above, volumes of solids of revolution computed using the washer/disc method are special cases. They are easier in the sense that the cross sections are always discs or washers, so their areas look like $\pi R(x)^2$ (discs) or $\pi R(x)^2 - \pi r(x)^2$ (washers), where $R(x)$ is the radius of the (big) disc, and $r(x)$ is the radius of the smaller, removed, disc. Thus, in this case, the volume always looks like

$$\int_a^b \pi R(x)^2\ dx \quad \text{or} \quad \int_a^b (\pi R(x)^2 - \pi r(x)^2)\ dx.$$

(All this is expressed in the case where we are revolving about an axis parallel to the x-axis!)

So what's the analytical part? There's nothing to do except figure out the formulas for $R(x)$ and $r(x)$. These will be distances between points (x, y) on curves bounding the revolved region and the point (x, c) on the axis of revolution $y = c$. Your problem is to express the y as a function of x.

We've done so many problems in class and on quizzes, that I'm not going to do another example here (where I can't draw pictures, so would have to be confusingly verbose). I'll simply give a problem list.

Problems you should be able to do: p.322: 11-14, 15, 17, 19, 21, 23,27,29, 37, 39,41, 43, 45.

If you have the general reasoning down and have reached the point where you can reliably find the radii, don't waste time doing a zillion problems. If you STILL are goofing up the radii, come for help!

6.2 Volumes by Cylindrical Shells

I've just decided, being a kitchen sort of person, that if disc/washer is volumes by slicing, shells ought to be called volumes by peeling.

So we have a region R lying to one side of our axis of revolution, say $x = c$. If we were slicing our solid, we'd slice it *perpendicular* to the axis of revolution. This would amount to slicing our region R perpendicular to the axis of revolution (choosing a y value) into thin strips, and revolving those strips to create discs/washers (whose volumes we then "add up", for y ranging from the smallest y value in the region to the largest y value in the region). Now what if we slice our region R *parallel* to the axis of revolution (choosing an x value) into thin strips and revolve one of those strips about the axis. We get a *shell.* The mnemonic is that we add up the volumes of those shells, for x ranging from the smallest x value, say $x = a$, in the region to the largest x value, say $x = b$, in the region. That is to say,

$$V = \int_a^b (\text{Volume of shell at } x).$$

The next step is to note that

$$(\text{Volume of shell at } x) = 2\pi r(x)h(x)dx,$$

where $r(x)$ is the radius of the shell (using x midpoint between inner and outer walls), $h(x)$ is the height of the shell and dx is the thickness of the shell.

How hard is the radius to find? It is simply the distance between x and the axis $x = c$ of revolution, that is, $r(x) = |x - c|$, ALWAYS. Of course, if our region lies to the positive side (right side, in this case) of the axis of revolution, then $x \geq c$, so $r(x) = x - c$. If our region lies to the negative side (left side, in this case) of the axis of revolution, then $x \leq c$, so $r(x) = c - x$. The short, informal mnemonic is, remember, "distance is larger - smaller."

How hard is the height $h(x)$ to find? Well, this depends on the problem. It will be the distance between two points (x, y_1) and (x, y_2) on the boundary of the region. Since these two points have the same x-coordinate, that distance is the same as the distance between y_1 and y_2, which, in turn, is the larger of y_1 and y_2, minus the smaller of them (or $|y_1 - y_2|$, for short!). So the problem comes down to finding y_1 and y_2 (as functions of x!) and knowing which one is larger.

In sum, we have

$$V = \int_a^b 2\pi r(x)h(x)dx = \int_a^b 2\pi|x - c| \cdot |y_1(x) - y_2(x)|dx$$

in general, and our problem is always to determine $y_1(x)$ and $y_2(x)$ from the description of the region.

Example. Look at number 36 (picture) on p. 330. Suppose we revolve about $x = -2$. Then $r(x) = |x-(-2)| = x+2$ (since the region lies to the right — positive — side of $x = -2$. Our height will be the distance between the points (x, x^2) and $(x, -x^4)$. Since x^2 is the larger and $-x^4$ is the smaller of these two y-values, then, this distance is $h(x) = x^2 - (-x^4) = x^2 + x^4$. The smallest x-value in the region is 0, the largest, 1. These give our limits of integration. Thus, the volume would be

$$V = \int_0^1 2\pi r(x)h(x)dx = \int_0^1 2\pi(x + 2)(x^2 + x^4)dx.$$

Now before reading further, do the same problem with the axis of revolution being $x = 3$. Your answer should be

$$V = \int_0^1 2\pi r(x)h(x)dx = \int_0^1 2\pi(3 - x)(x^2 + x^4)dx.$$

Example. Look at number 24 (picture) on p. 329. Let's revolve about $y = 3$ and do shells. We're slicing the region parallel to $y = 3$, so choosing y values. Everything we write must now be expressed in terms of y. The smallest and largest y values in the region are 0 and 2, respectively. These give our limits of integration, so we have

$$V = \int_0^2 2\pi r(y)h(y)dy,$$

for starters. Continuing, since our axis of revolution is $y = 3$,

$$r(y) = |y - 3| = 3 - y.$$

(Why the last equality? Because our y values in the region are smaller than 3.) The height $h(y)$ of the cylinder will be the distance between the points $(y^2/2, y)$ and $(y^4/4 - y^2/2, y)$, that is,

$$h(y) = |y^2/2 - (y^4/4 - y^2/2)| = y^2/2 - (y^4/4 - y^2/2) = y^2 - y^4/4,$$

the next to last equality holding because (look at picture) $y^2/2$ is larger than $y^4/4 - y^2/2$ when y lies in the interval $[0, 2]$. Thus, our volume is

$$V = \int_0^2 2\pi r(y)h(y)dy = \int_0^2 2\pi(3-y)(y^2 - y^4/4)\ dy.$$

Now do the same problem with the axis of revolution as $y = 0$. Your answer should be

$$V = \int_0^2 2\pi r(y)h(y)dy = \int_0^2 2\pi y(y^2 - y^4/4)\ dy.$$

Problems you should be able to do: p.328 1,3,9,11,17,21,23,24

Finally, you may have problems where you need to choose washer or shell, based on what is feasible or what is easier. You should practice choosing.

Problems you should be able to do: p.329: 25, 26 (c. should be x=8 and d. should be y=4), 27,29,31,33,34,36.