# KNOWING, UNDERSTANDING, OWNING and PLAYING

While mastery of computational algorithms (e.g. computing areas, volumes of solids of revolution, derivatives, integrals, limits) is expected of you, and will be tested on exams, the wider aim of the course is to develop a deeper familiarity with and understanding of the topics treated.

Computational mastery is a well-defined aim that most of you can achieve by the obvious means of conscientious practice. Moreover, whether you have achieved this mastery is easily verifiable — by either me or you — by the simple means of setting representative problems to which the solutions are accessible and seeing whether you can arrive at those solutions reliably and efficiently. (The notion of mastery includes both that of reliability and that of efficiency. Do not consider yourself out of the woods if you can do problems correctly MOST of the time, nor if you can do them correctly EVENTUALLY, i.e. given enough time.)

The wider aim of "understanding" is more difficult to define, to achieve and to evaluate. It includes, absolutely, knowing the precise meaning of all terms (e.g. what the natural logarithm function IS, not just what computational laws it obeys), knowing the precise statements of all theorems that we have available for use (e.g. the Mean Value Theorem for Integrals, the Intermediate Value Theorem, the Laws of Exponents), but is not limited to this knowledge. (I can teach my 8th grade children or my mother the definitions and statements of the theorems of the course, just as I can teach them to recite poetry in Russian. But this constitutes neither "understanding calculus," in the former case, nor "speaking Russian" in the latter.)

Being able to state a theorem is a *prerequisite* to understanding it. (You can't claim to "understand" an entity you can't identify!) Understanding a theorem is a *prerequisite* to "owning" it — having the ability to use it (apply it) to solve problems. It is this last that is your goal. All else is steps toward it. To illustrate "knowing (the statement of)" versus "understanding" versus "owning," I offer the following example drawn from your recent experience.

- I have a continuous function $f$ that has a zero (call it $a$) and whose limit at infinity is infinity. Because I "own" the Intermediate Value Theorem and the definition of a limit at infinity's being infinity, I am able to deduce that for any positive $b$, the equation $f(x) = b$ has a solution in $(a, \infty)$, i.e. that the range of $f$ on $(a, \infty)$ contains $(0, \infty)$. (We will use just this argument very soon to see that $\ln(x) = 1$ has a solution in $(1, \infty)$, which we then denoted $e$!). How do we deduce this? Fix $b > 0$. Because I understand what $\lim_{x\to\infty} f(x) = \infty$ *means* (look it up and study it if you don't!), I know that there must be some $c > a$ such that $f(c) > b + 1$. (The only important thing about $b + 1$ here is that it is a positive number bigger than $b$.) Because I understand the IVT and know that $f$ is continuous, I know that since $f(a) < b < f(c)$ (recall $f(a) = 0$ and $f(c) > b + 1 > b > 0$), there is $d$ in $(a, c)$ such that $f(d) = b$. Such a $d$ constitutes a solution to $f(x) = b$ in $(a, \infty)$, as desired.

  I point out here that "knowing" (being able to state) the IVT and the limit definition, and even being able to compute limits at infinity, while they might have gotten me through my Calc I exam on the subjects, were not enough to make them available to me as problem-solving tools. I had to understand their meanings so I could apply them to my particular case, and I had to have enough *experience* with them so that I would *think* of applying them to the situation at hand. This last is what I mean by "owning" them.

Now the goal of whoever taught you Calc I was — or should have been — both to bring you to mastery of various computational algorithms for limits, derivatives and integrals, and to help you come to *own* the material of that course — which is assumed to have included various notions of limit, continuity, rates of change and the derivative, sign analysis of derivatives and its graphical implications, MVT, IVT, the definite integral, FTC 1 and FTC 2.

If your class was typical (if not optimal) your exams consisted mostly of rather straightforward applications of whatever computational techniques had just been learned. Because of this, you may have slipped into the habit of ignoring the "theory," as I've often heard students call the understanding/ownership aspects, and shrugging off the more complex or difficult homework problems. If this is the case, then you will now find yourself panicked and perplexed when, assuming that you own the Calc I material, I casually apply the IVT or MVT and definitions you never quite understood, as in the illustration above, or expect you to do a nontrivial integral that chains together all the techniques you ever learned. This seems totally unfair. Why didn't someone warn you? Prepare you? Well, they probably did — or tried.

If you go back to your Calc I assignments, you'll find all these gruesomely complex or abstract problems — numbered 70 and above, so you *knew* they wouldn't be on the exam — that just seemed like gratuitous torture at the time, but in fact constituted your instructor's attempt to raise your understanding and computational mastery to the level of ownership you would need in the real world — "the real world" meaning future contexts, be they succeeding mathematics courses or work in your chosen field of study, where the material under consideration is actually applied to solve problems. Now, all of a sudden, those gruesome problems are just passing applications of previously learned material carried out in the midst of coming to grips with new material. (Indeed, where do you think we get gruesome problems to give to our algebra/trig students? We pluck them from the midst of everyday

Calc I problems, e.g. simplifying a difference quotient in preparation for taking a limit, simplifying and factoring a derivative in preparation for doing a concavity analysis. Similarly, we get gruesome limit problems for our Calc I students by rifling through subsequent subjects for those sorts of limits that arise frequently. Problem sets in any decent mathematical text are designed along precisely these lines. *Remember: You're learning this material for a reason, and we, your instructors and textbook writers know what it is. All our choices of examples and problems are chosen with this knowledge in view.*)

Unfortunately, the harsh reality is that one can never truly fathom the depth at which one will need to have mastered the material until one is actually faced with the situations in which it is used. Do you think that a fellow who's never stepped out of the Chicago city limits would train physically and pack the same supplies for his first month-long wilderness trek as he would for his second? Even if he has an experienced friend who gives him advice beforehand, he's still not truly going to appreciate it and prepare himself as well as he will after his *own* experience. The point is, everybody has to have a first trek. If you're wise, you listen to your experienced friend and pack that mysterious collection of supplies even if you don't know why you will want Benadryl, a big sheet of plastic, and a small garden spade, and you'll let him choose your hiking boots. Now this analogy falls apart at the point where one notes that your hiker friend can probably explain to you in a reasonable amount of time the specific reasons for each piece of his advice. (E.g. the Benadryl is for poison ivy, nettles and the hundreds of bug bites you'll get even though you did bring DEET.) In our context, it is usually simply impracticable to do more than hint at why on earth you need to own this theorem or definition, why you need to master this hideous technique when all the problems you've met so far can be done in an "easier" way.

Part of the issue is time. We don't have it. Part is that in order to describe the situations that will justify your investment of time and effort, we would have to introduce ideas and terms that you don't have the tools to understand yet. Imagine, if you will, explaining to your kid sister who's taking high school algebra why her teacher is making her practice simplifying what are in effect complex difference quotients. To say anything remotely meaningful, you'll have to introduce the notion of derivative ... oh, wait, for that to have any meaning, she's got to know what a limit is ... oh, wait ... Hey, sis, just take my word for it, you'll be glad you got good at this stuff. This is basically what we, your instructors, have to do much of the time: Say, "take my word for it, you'll need it, I'm not just torturing you."

It's fairly easy to take our word for it on computational matters. It's less easy on matters of "understanding" and "owning," where we can't give you a well-defined, concrete goal and a means for you to determine whether you have attained it. You must, I'm afraid, simply *trust* that we are pushing you in the direction and at the speed we're pushing you because we know what's on the other side of the hill in the wilderness.

So, practically speaking, how can you reach the stage of "owning" a new definition or theorem? There's not a one-size-fits-all procedure for this, but the essential elements are *contemplation* and *experience*. Mathematicians, and others who make their living working in abstract settings, all share a notion of "playing with" a new idea. Faced with a new definition or theorem that they know will be important to them, they begin the quest for ownership by studying it until it is no longer just words, but a statement with meaning. (This may involve reviewing the meanings of terms included in the definition or theorem.) Then they will start playing. They will search through their experience for examples of objects and situations that satisfy the definition or the hypotheses of the theorem and in the latter case verify that the conclusion holds. They will also search for things that do *not* satisfy the definition or do not satisfy the hypotheses of the theorem, in the latter case, trying to see how the conclusion of the theorem can fail. Learning what an object *is not* is part of learning what it *is*. If the new definition is well known and understood by others, he will pick their brains, will take their experience for his own by reviewing their published conclusions on the subject — examples of objects satisfying the definition, work showing that certain objects do not satisfy the definition, theorems showing consequences of satisfaction of the definition. (E.g. Suppose my new definition is *continuity*. I would try to construct or find in the literature examples of continuous functions and discontinuous ones. I would look for theorems of consequence that have continuity as a hypothesis to give me some idea of what continuity buys me and why I should be interested in it in the first place.)

Now, of course, we do not expect you to do a literature search every time you meet a new definition. We do not expect you, either, to have mastered the art of "playing." What we do is push you into a sort of "guided playing" by assigning as exercises the sorts of things we would do to get to know a new idea. So, for instance, when you first met the definition of continuity, we made you show that various functions were continuous, show that others were not, show that if a function was continuous, these nice things would happen. This was meant to be "guided playing." Unfortunately, we often neglect to tell you this, so these exercises seem to *you* to have the purpose of testing your knowledge or, if you're in a bad mood, simply killing your time. When this is so, you tend to do these exercises with your brain turned off, so that what was meant to be a useful experience did, indeed, turn into a pointless waste of time. From here on out, I hope you will look at such exercises in the right light, with your brain turned on and a purpose in mind. If you do, the results of your expenditure of time should be much more valuable.