**The notion of domination**

In determining convergence/divergence of improper (at $\infty$) integrals and infinite series, we have to do a lot of tests of the form

$$\lim_{x\to\infty} f(x)$$

It is important that we develop the skills to compute such limits efficiently. This requires both a solid understanding and reliable application of the basic tools, such as l'Hopital's rule, and the ability to recognize what the outcome would be if one used those tools repeatedly (without necessarily doing so). For instance, we should be able to recognize that

$$\lim_{x\to\infty} \frac{(\ln(x))^{37}}{x^{2/3}} = 0$$

because we understand that if we use l'Hopital's rule 37 times, we'll end up with our original limit equal, for some $c \neq 0$, to $\lim_{x\to\infty} \frac{c}{x^{2/3}}$. We should both know the result and remember how we could get there explicitly if we cared to write the thing out. (An analogue: What is the 23rd derivative of $14x^{22} + 15x^{11} - .459x^5$? Do you need actually to carry out all those derivatives to find the answer? No, because you know what happens to the degree of a polynomial when you differentiate it once.)

First, I'm going to carefully establish some terminology, so that we won't start making mistakes when we get to complicated situations. (Informality is nice. Intuition is nice. But it needs to be solidly grounded.)

$$g(x) >> f(x) \text{ (read "}g(x)\text{ dominates }f(x)\text{") at } \beta \text{ means } \lim_{x\to\beta} \frac{f(x)}{g(x)} = 0$$

As with $>$ and $<$, $g(x) >> f(x)$ and $f(x) << g(x)$ mean the same thing.

Thus, e.g., $x^2$ dominates $x^3$ at 0 since $\lim_{x\to 0} \frac{x^3}{x^2} = 0$, while $x^3$ dominates $x^2$ at $\infty$, since $\lim_{x\to\infty} \frac{x^2}{x^3} = 0$.

Some properties of $<<$:

1. If $f(x) << g(x)$ and $g(x) << h(x)$ then $f(x) << h(x)$. (*Think it through.*)
2. If $f(x) << g(x)$ at $\beta$, then $\lim_{x\to\beta} \frac{|g(x)|}{|f(x)|} = \infty$. (*Why do we need the absolute values?*)
3. If $f(x) << g(x)$ at $\beta$ then for $x$ sufficiently close to $\beta$, $|f(x)| < |g(x)|$. (*Again, why do we need the absolute values? Why only eventually?*)
4. $f(x) << g(x)$ iff $|f(x)| << |g(x)|$, and multiplying either or both sides by a NONZERO constant does not change dominance. (*Think it through.*)

Note that $<<$ is about *size* in the limit, not who is numerically greater, so that, e.g. at $\infty$, even though $-x^4 \leq x^2$ eventually, in fact $-x^4$ *dominates* $x^2$, i.e. $-x^4 >> x^2$, i.e. $\lim_{x\to\infty} \frac{x^2}{-x^4} = 0$. If you are comparing nonnegative functions, this annoying problem disappears. See the third property above.

Note that $0 < f(x) < g(x)$ (eventually) is not a strong enough condition to ensure $f(x) << g(x)$. For instance:

$$x^2 < x^2 + 3 \text{ for all values of } x, \text{ but } \lim_{x\to\infty} \frac{x^2}{x^2+3} = 1 \neq 0 \text{ so } x^2 \not<< x^2 + 3.$$

Sure, $x^2 + 3$ is "bigger than" $x^2$ near infinity, but not enough bigger for dominance. Same goes for $5x^2$. All these guys are "about" the same size. None of them dominates the other. I won't introduce a formal notation for this, but will continue saying, informally, that

$$f(x) \text{ and } g(x) \text{ "are about the same size," or "look about alike" (at } \infty) \text{ when } \lim_{x\to\infty} \frac{f(x)}{g(x)} = k \text{ for some NONZERO } k$$

and that

$$f(x) \text{ and } g(x) \text{ "are the same size," or "look alike" (at } \infty) \text{ when } \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

This is my own informality, but I think it helps keep the picture straight. If two guys are about the same size, neither dominates the other. If two guys are about the same size, they dominate the same other guys, and are dominated by the same other guys.