General Notes on Indeterminate Forms

Throughout this discussion, the symbol β may be replaced by any choice of real number, or by ∞ or by $-\infty$. This will allow us to write several analogous statements as one. (For instance, I may write truly that if $\lim_{x\to\beta} f(x) = \infty$ and $\lim_{x\to\beta} g(x) = p > 0$ then $\lim_{x\to\beta} f(x)\cdot g(x) = \infty$.) The proofs in the cases where β represents a real number, β represents ∞ and β represents $-\infty$ would be similar, but independent proofs; but the statement holds whichever symbol replaces β.

Recall that knowing that, say, $\lim_{x\to\beta} f(x) = \infty$ and $\lim_{x\to\beta} g(x) = \infty$ is *insufficient information to determine* the behavior of $\dfrac{f(x)}{g(x)}$ as $x \to \beta$. The shorthand for this statement is

$$\text{“}\tfrac{\infty}{\infty} \text{ is an indeterminate form.”}$$

I loathe this shorthand, because people are often tempted to interpret the expression $\frac{\infty}{\infty}$ as “infinity divided by infinity,” which is nonsense: ∞ is not a number, let alone something one can divide by or into.

Nevertheless, for the sake of brevity and mnemonics, I will list the indeterminate forms in this fashion, with the proviso that they are simply visual reminders of statements such as the one above. **Never, ever** treat these symbols as numbers. In particular, writing something like

$$\text{“} \lim_{x\to\infty} \frac{x^2+1}{x^2-3} = \frac{\infty}{\infty}\text{”}$$

is a solecism of the worst sort, and I will not allow it. You mean to say, but are too lazy to do so, that

$$\text{both } x^2+1 \to \infty \text{ and } 4x^2-3 \to \infty \text{ as } x \to \infty.$$

Some familiar indeterminate forms:

$\dfrac{\infty}{\infty}$, $\dfrac{0}{0}$, $0\cdot\infty$ (and variants with $-\infty$ replacing one or more ∞)

$\infty - \infty$, aka $\infty + (-\infty)$)

[There are other forms that are indeterminate in more delicate ways. E.g., $\dfrac{1}{0}$ is indeterminate, while its signed versions $\dfrac{1}{0^+}$ and $\dfrac{1}{0^-}$ are determinate. Huh? Compare, say,

$$\lim_{x\to 3} \frac{1}{x-3} \quad (d.n.e.); \quad \lim_{x\to 3^+} \frac{1}{x-3} = \infty; \quad \lim_{x\to 3^-} \frac{1}{x-3} = -\infty.$$

Knowing that $g(x) \to 0$ is not quite enough to determine the behavior of $\frac{1}{g(x)}$. Knowing that $g(x) \to 0^+$ or $g(x) \to 0^-$ *is* enough.

Power forms

We have had cause before to deal with limits of the form

$$\lim_{x\to\beta} f(x)^q, \qquad q \text{ rational.}$$

What this limit is depends on $\lim_{x\to\beta} f(x)$ and on the value of q, but we have the arithmetic to handle it.

We have now generalized our notion of exponentiation and so can reasonably define and analyze functions of the form

$$f(x)^{g(x)},$$

at least, where the base function $f(x)$ is nonnegative. (We still do not have, and will *never* have, a meaning for, say, $(-2)^c$, when c is irrational.)

The first thing we must do is determine the arithmetic of limits of such functions. The basic theorem (see proof in text) is the following:

Theorem. Suppose that $g(x)$ and $f(x)$ are defined, and $f(x) \geq 0$ near β.

1. If $\lim_{x\to\beta} f(x) = a$ and $\lim_{x\to\beta} g(x) = p$, then
$$\lim_{x\to\beta} f(x)^{g(x)} = a^p,$$
provided a and p are not both equal to 0.

   That is,
$$\lim_{x\to\beta} f(x)^{g(x)} = \left(\lim_{x\to\beta} f(x)\right)^{\lim_{x\to\beta} g(x)},$$
provided both limits exist (as real numbers!!!) and they are not both equal to 0.

2. If $\lim_{x\to\beta} f(x) = \infty$ and $\lim_{x\to\beta} g(x) = \infty$, then $\lim_{x\to\beta} f(x)^{g(x)} = \infty$.

3. If $\lim_{x\to\beta} f(x) = b \neq 1$ and $\lim_{x\to\beta} g(x) = \infty$ then
$$\lim_{x\to\beta} f(x)^{g(x)} = \lim_{x\to\beta} b^{g(x)} = \lim_{u\to\infty} b^u.$$
This limit will, of course, be 0 if $0 < b < 1$ and will be ∞ if $b > 1$.

[Note that all of these limits are the intuitively obvious ones. You can digest them as, e.g. "if $f(x)$ is getting big, and $g(x)$ is getting big, then $f(x)^{g(x)}$ is getting big," which is probably how you came to terms with earlier determinate forms. This kind of stuff isn't good enough to **prove** a result, but it sure helps remember one.]

Part 1 of the theorem tells us that b^p is determinate if b and p are not both 0, i.e. if b^p is defined. Part 2 tells us that ∞^∞ is determinate, and part 3, that b^∞ is determinate if $b \neq 1$. Now let's deal with the loose ends. First, what about $\infty^{-\infty}$ and $b^{-\infty}$? To do these, simply rewrite $f(x)^{g(x)}$ as $\frac{1}{f(x)^{-g(x)}}$ and apply the theorem with $-g(x)$ replacing $g(x)$. Do this RIGHT NOW, and see what you get.

Finally, what about, 1^∞, ∞^0 and 0^0? These are

New (to you) indeterminate forms: 1^∞, ∞^0 and 0^0.

With the old indeterminate forms, we learned many techniques for rewriting the argument of the limit into a new algebraic form that we knew how to deal with. Which technique applied depended on the form and who β was. The way we deal with indeterminate power forms will not depend on which of the three we have, and will not depend on β. Shortly put, it is to rewrite
$$f(x)^{g(x)} = e^{g(x)\ln f(x)},$$
calculate
$$\lim_{x\to\beta} g(x)\ln f(x)$$
and then go back and use our answer to calculate
$$\lim_{x\to\beta} f(x)^{g(x)} = \lim_{x\to\beta} e^{g(x)\ln f(x)}.$$

Of course, nothing is this easy. If $f(x)^{g(x)}$ is an indeterminate power (1^∞, 0^0 or ∞^0), then $g(x)\ln f(x)$ will be an indeterminate *product* ($\infty \cdot 0$, $0 \cdot (-\infty)$ or $0 \cdot \infty$, respectively). But indeterminate products can be rewritten, in turn, as indeterminate quotients, and l'Hopital's Rule chews these up and eats them for breakfast. It's fun, actually. Just wait and see.