Review of some important limit theorems

I am assuming here and in class that you have a basic working knowledge of the meaning of and arithmetic of limits, both as $x \to a$ for real numbers a and as $x \to \infty$ and of continuity. Some of you will probably need to review the basics, most certainly those of you who took Calc I earlier than this spring. You will want to reread sections 2.2-2.6 of the text. At the minimum, you should easily be able to do the following sorts of problems (try some to check!): 2.2: 11-54, 2.4: 11-18, 21-56, 57-62, 2.5: 1-26, 2.6: 13-33.

What I gather below are the important nonarithmetic theorems on limits that we will need in the coming weeks, together with the special trig limits that you may have forgotten even the existence of.

Here and elsewhere in limit work, I will use α, β, γ to represent a "constant" that may be real, one-sided real (i.e. c^+ or c^-) or may be either of ∞ or $-\infty$.

If α is real, "near α" means "on some open interval containing α. If $\alpha = \infty$, "near α" means " on some open interval (a, ∞)," and if $\alpha = -\infty$, "near α" means " on some open interval $(-\infty, a)$.

1. **Theorem. [The Sandwich (or Squeeze) Theorem for Limits]** If
$$f(x) \leq g(x) \leq h(x)$$
near α and
$$\lim_{x\to\alpha} f(x) = \beta = \lim_{x\to\alpha} h(x),$$
then also
$$\lim_{x\to\alpha} g(x) = \beta.$$

2. **Theorem. [The Substitution Theorem for Limits]** If
$$\lim_{x\to\alpha} f(x) = \beta$$
and
$$\lim_{u\to\beta} g(x) = \gamma$$
then
$$\lim_{x\to\alpha} g(f(x)) = \gamma.$$
[Why is this called "substitution"? It says that if $f(x) \to \beta$ as $x \to \alpha$ then we may substitute u for $f(x)$ and $u \to \beta$ for $x \to \alpha$ in a limit:
$$\lim_{x\to\alpha} g(f(x)) = \lim_{u\to\beta} g(u).$$

3. **Corollary. [Continuity of Composites of Continuous Functions]** If f is continuous at a and g is continuous at $f(a)$ then $g \circ f$ is continuous at a.

4. **Theorem.** If $f(x) \leq g(x)$ near α, then
   - if $\lim_{x\to\alpha} f(x)$ and $\lim_{x\to\alpha} g(x)$ exist, then $\lim_{x\to\alpha} f(x) \leq \lim_{x\to\alpha} g(x)$;
   - if $\lim_{x\to\alpha} f(x) = \infty$ then $\lim_{x\to\alpha} g(x) = \infty$;
   - if $\lim_{x\to\alpha} g(x) = -\infty$ then $\lim_{x\to\alpha} f(x) = -\infty$.

5. **Theorem. [Special Trigonometric Limits]**
$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1, \qquad \lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1, \qquad \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0.$$

[Recall that we had to establish the first and third of these limits in order to prove that $\frac{d}{d\theta}\sin\theta = -\cos\theta$ and $\frac{d}{d\theta}\cos\theta = \sin\theta$.]