Study Guide for 7.1: Inverse functions and their derivatives

FOR STARTERS: Make sure you read the "What to Memorize" page, and do as it says.

7.1: Inverse Functions

This section is really just a background tool section for us, so that we may efficiently study the inverses ($e^x, \sin^{-1} x, \tan^{-1} x$ and $\sec^{-1} x$) of the important transcendental functions $\ln x, \sin x, \tan x$ and $\sec x$). It's *using* the material that you want to get good at. What are the main points?

1. When is a function invertible on a domain? When it is one-to-one on the domain. Do you *really* understand what "one-to-one" means? What tools do you have for determining whether a function is one-to-one on its domain? Answer: "one-to-one" is a consequence of "increasing" and of "decreasing." That is, if a function is increasing or if it is decreasing on its domain, then it is one-to-one on the domain. Determining that a function is increasing or decreasing on a domain is, in turn, something we have a tool for: the sign of the first derivative. If you have forgotten how to use it, review section 4.3 of the text. (Recall, e.g., that we showed that $\ln x$ must be increasing, hence invertible, on its domain of $(0, \infty)$ by noting that its derivative, $\frac{1}{x}$, is positive on that domain. You might be expected make such an argument for a different function.)

2. What are the combined properties of an invertible function f and its inverse g?
   (a) The domain of one is the range of the other.
   (b) For x in the domain of f, the equations $y = f(x)$ and $g(y) = x$ are equivalent to one another. (Thus, e.g. $y = \ln x$ is equivalent to $exp(y) = e^y = x$ and $y = x^3$ is equivalent to $\sqrt[3]{y} = x$.)
   (c) For all x in the domain of f, $g(f(x)) = x$, and for all y in the domain of g, $f(g(y)) = y$. (Thus, e.g. $e^{\ln(x)} = x$ for all x in $(0, \infty)$ and $\ln(e^y) = y$ for all y in $(-\infty, \infty)$. Similarly, for all x and y, $\sqrt[3]{x^3} = x$ and $(\sqrt[3]{y})^3 = y$.)
   (d) The graphs of f and g are reflections of one another across the line $y = x$. (Thus, e.g., we used our graph of $y = \ln x$ to immediately produce the graph of $y = e^x$ as its reflection across the line $y = x$. We also, in problem 10, 7.1, used the graph of $y = \tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ to produce the graph of its inverse, $y = \tan^{-1} x$, which we shall study in some detail in 7.4.).
   (e) To find $g'(x)$, given f', we used implicit differentiation with respect to x. We said:

$$y = g(x) \quad iff \quad f(y) = x,$$

   differentiated the latter implicitly to get $f'(y)\frac{dy}{dx} = 1$, from which

$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(g(x))}.$$

   That is, in the notation $g(x) = f^{-1}(x)$,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))},$$

   or equivalently, writing $x = f(u)$,

$$(f^{-1})'(f(u)) = \frac{1}{f'(u)}.$$

   We used this in 7.3 to show, e.g., that $\frac{d}{dx}e^x = e^x$. You should be able to reproduce this argument, as well as to use the "formula" for finding $f^{-1}(a)$ at a point (a, b), given a formula for $f(x)$, as you did in the homework exercises.