Study Guide for 7.2, 7.3: Logarithmic and Exponential Functions

There's a lot of material in these two sections. You'll need to study from two angles:

- understanding based on what $\ln x, \log_a x, e^x, a^x$ *mean*, i.e. on how they are defined;
- computational proficiency based on the basic formulas and facts that we proved from the definitions.

Let's begin with the first bit of meaning. We recalled that FTC Part 1 tells us:

"If f is continuous on an interval I, then f has an antiderivative on I. In particular, for any element a of I, the function

$$F(x) = \int_a^x f(t)\ dt$$

is an antiderivative of F on I. (To wit, it is the antiderivative F satisfying $F(a) = 0$.)"

Next, we noted that the function f defined by $f(x) = \frac{1}{x}$ is continuous on $(0, \infty)$, so for any $a > 0$, the function $\int_a^x \frac{1}{t}\ dt$ is an antiderivative of $f(x)$ on $(0, \infty)$. We chose to name the particular one whose value is 0 at $x = 1$. We called this function the natural logarithm, and denoted it ln. That is, we **defined**

$$\ln x = \int_1^x \frac{1}{t}\ dt \quad \text{on } (0, \infty).$$

It is an immediate consequence of the definition that $\ln x$ is the unique function defined on $(0, \infty)$ satisfying

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad \text{and} \quad \ln(1) = 0.$$

Indeed, we might have taken this as the definition (once we knew such a function existed) and everything we know about ln can be proved from these two bits of information.

So what *do* we know from this, and how did we get it?

1. We know that
$$\ln(xy) = \ln(x) + \ln(y) \quad \text{for } x, y > 0.$$
Why? We proved that for $a > 0$, the two functions $\ln(ax)$ and $\ln(a) + \ln(x)$ must be identical on $(0, \infty)$ by showing they have the same derivative there, and the same value at 1. From this basic equation, we proved
$$\ln(\frac{1}{y}) = -\ln(y) \quad \text{for } y > 0$$
by writing $0 = \ln(1) = \ln(y\frac{1}{y})$ and applying the first fact with $x = \frac{1}{y}$. With a little more work, we used the first fact to show that
$$\ln(x^q) = q\ln(x) \quad \text{for } x > 0, \ \ q \text{ any rational.}$$
(Actually, we only did it in class for q a positive integer, but you can look at the more general case in the text.)
Remember that the only real result here is that multiplication inside the natural logarithm turns into addition outside the logarithm. It is easy, then, to remember that division (the inverse operation of multiplication) inside the log turns into subtraction (the inverse operation of addition) outside the log, and taking to a power n (multiplication of x by itself n times) inside the log turns into multiplication by n (adding $\ln x$ to itself n times) on the outside.

2. We know that
$$\frac{d}{du}\ln u = \frac{1}{u} \quad \text{on } (0, \infty),$$
and hence,
$$\int \frac{1}{u}\ du = \ln u + C \quad \text{on } (0, \infty).$$
Why? Well, this is just immediate from the definition. But we also showed that on $(-\infty, 0)$, the function $\ln(-u)$ is defined, with derivative $\frac{1}{u}$, which gives us

$$\frac{d}{du}\ln(-u) = \frac{1}{u} \qquad \text{on} \;\; (-\infty, 0),$$

and hence,

$$\int \frac{1}{u}\, du = \ln(-u) + C \qquad \text{on} \;\; (-\infty, 0).$$

We slapped these together as the formulas

$$\frac{d}{du}\ln|u| = \frac{1}{u} \qquad \text{on} \;\; (0,\infty) \;\; \text{and on} \;\; (-\infty, 0),$$

and

$$\int \frac{1}{u}\, du = \ln|u| + C \qquad \text{on} \;\; (0,\infty) \;\; \text{and on} \;\; (-\infty, 0).$$

3. We know that $f(x) = \ln x$ has derivatives $f'(x) = \frac{1}{x}$ and $f''(x) = \frac{-1}{x^2}$, which are, respectively, positive and negative on all of $(0, \infty)$, so f is increasing and concave down on $(0, \infty)$.

   Moreover, we have that

   $$\lim_{x\to\infty} \ln(x) = \infty$$

   and

   $$\lim_{x\to 0^+} \ln(x) = -\infty.$$

   We didn't get around to proving this first limit in class, but it's pretty easy: We know our function is increasing, so to show that its limit at ∞ is ∞, all we need is to show that for any $L > 0$, there is some b such that $\ln(b) \geq L$. So fix such an L. Now choose any old $c > 1$. Since $\ln x$ is increasing, and $\ln(1) = 0$, we have $\ln c > 0$. Let n be any integer bigger than $L/\ln(c)$, so that $n\ln(c) > L$. Then $\ln(c^n) = n\ln(c) > L$, so c^n is our desired b.

   We can prove the second limit easily from the first: Since $\frac{1}{x} \to \infty$ as $x \to 0$, we have

   $$\lim x \to 0^+ \ln(x) = limx \to 0^+ - \ln(\frac{1}{x}) = -\infty.$$

   Cute, huh?

   Note that since $\ln(x)$ attains arbitrarily large positive and arbitrarily large negative values, and is continuous (differentiability implies continuity), by the IVT (reload!) it must attain every real value. That is, $range(\ln) = (-\infty, \infty)$.

   Now once we have all of this, we know exactly what the graph of $f(x) = \ln(x)$ looks like. You should have a picture of it immediately accessible in your head to reason from.

Now, let's recall the definition of $y = exp(x)$. It is the inverse of the natural logarithm function $y = \ln(x)$. This *immediately* gives us its graph (since we paid attention in Section 7.1), including its domain, $(-\infty, \infty)$, and its range, $(0, \infty)$, from which we can do a great deal of reasoning. (E.g. What is $\lim_{x\to-\infty} exp(x)$? What are the zeroes, if any, of $exp(x)$? When is $exp(x) > 1$, when is $exp(x) = 1$, when is $exp(x) < 1$?)

We used implicit differentiation of the equation $\ln(y) = x$ to prove that exp is its own derivative. You should be able to reproduce this.

We showed that the arithmetic laws for ln translated into the following laws for its inverse function exp:

$$exp(u + v) = exp(u) \cdot exp(v) \qquad \text{for all} \;\; u, v,$$

$$exp(-v) = \frac{1}{exp(v)} \qquad \text{for all} \;\; u$$

and

$$exp(qu) = [exp(u)]^q \qquad \text{for all } \ u.$$

These may be easily remembered as "addition inside *exp* turns to multiplication outside *exp*" — the reverse of what happens with its inverse, ln.

We defined the number e to be the solution of $\ln x = 1$, which also makes, equivalently, $e = exp(1)$. We used (and you should be able to use) the third law above to extend this fact to the identity

$$e^q = exp(q) \qquad \text{for any rational } \ q.$$

This allowed us, without ambiguity, to change our notation for $exp(x) = \ln^{-1}(x)$ to the more familiar and suggestive

$$e^x = exp(x).$$

(We had to know, e.g. that $exp(2)$ did not give a different value from $e^2 = e \cdot e$, or the notation wouldn't have been reasonable.) Our three laws above now become the more familiar laws of exponents:

$$e^{u+v} = e^u e^v, \quad e^{-v} = \frac{1}{e^v} \quad \text{and} \quad [e^v]^q = e^{qv} \quad (q \ \text{ rational}).$$

The inverse identities for ln and *exp* now become

$$\ln(e^x) = x \qquad \text{for } \ \text{all} x$$

and

$$e^{\ln x} = \ln x \qquad \text{forall} x > 0$$

and we have

$$e^0 = 1, \qquad e^1 = e.$$

Now we have a meaning for e^x for every (not necessarily rational) value of x, but for other positive values a besides e, a^x only has meaning when x is rational. (E.g. $5^{2/3}$ means "the square of the cube root of 5".) We extend the definition of a^x to x any real, as we extended the definition of e^x by *using* our extension for e^x as follows:

$$a^x =_{\text{def}} e^{\ln(a) \cdot x}.$$

Chase through the definition and laws of exponents for e to get the analogous laws of exponents for any positive base a. [In case you want to know, though I didn't do it in class, we finally have the full version of $(a^x)^y = a^{x \cdot y}$ without requiring that y be rational. The proof goes like this:

$$(a^x)^y = (e^{x \ln a})^y = e^{\ln(e^{x \ln a}) \cdot y} = e^{(x \ln a) \cdot y} = e^{(\ln a) \cdot (xy)} = a^{xy}.$$

Whew! Ick. But it's done.]

Mostly, you will be able to forget where our "new" definition of a^x came from and just whistle along using the laws of exponents. But we still need to know the calculus of $y = a^x$. You need the definition for this. Differentiate using the definition to get

$$\frac{d}{dx}(a^x) = (\ln a) a^x,$$

and the complementary antidifferentiation formula

$$\int a^x \, dx = \frac{a^x}{\ln a} + C.$$

To complete the picture, we define $\log_a(x)$ to be the inverse of the function a^x, but we really don't use these functions all that much, since one line of work shows that

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

and we can do anything we want with $\log_a(x)$ using this translation.

This is all I can write for now. I'm getting punchy. To wrap up: What is on these pages is what you should *know*, i.e. *understand* about who these functions are and what their properties are. What you need to do beyond this is to get computationally proficient with them. This is done by doing lots of exercises: integration, differentiation, logarithmic differentiation, solving equations, etcetera. If you can do all the computational exercises from the warmups and assigned problems fairly quickly and accurately without sweating, you're in good shape.