# MULTIVARIATE WAVELET FRAMES AND FRAME-LIKE SYSTEMS

M. Skopina

Department of Applied Mathematics and Control Processes
Saint Petersburg State University

April 11, 2010

## Tight frame in a Hilbert space $H$

$$\{f_n\} \subset H \ \forall f \in H \qquad \sum_n |\langle f, f_n\rangle|^2 = \|f\|^2$$

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R.J.Duffin and A.S.Schaeffer 1952

## Wavelet tight frame in $L_2(\mathbb{R}^d)$ with a dilation matrix $M$

$M$ is a $d \times d$ integer matrix whose eigenvalues are bigger than 1 in module;

$m := |\det M| > 1$,

$\psi_{jk}^{(\nu)}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d, \nu = 1, \dots, r$, $r \geq m - 1$

$\psi_{jk}^{(\nu)}(x) = m^{j/2}\psi^{(\nu)}(M^j x + k)$,

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## Wavelet frame decomposition

$$f = \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^d} \sum_{\nu=1}^{r} \langle f, \psi_{jk}^{(\nu)}\rangle \psi_{jk}^{(\nu)}$$

A tight wavelet frame $\{\psi_{jk}^{(\nu)}\}$ is said to have approximation order $n$ if there exist $\lambda > 1$ and $C > 0$ such that for any function $f$ in the Sobolev space $W_2^n$

$$\left\| f - \sum_{j<i} \sum_{k\in\mathbb{Z}^d} \sum_{\nu=1}^{r} \langle f, \psi_{jk}^{(\nu)}\rangle \psi_{jk}^{(\nu)} \right\|_2 \le C \frac{\|f\|_{W_2^n}}{|\lambda|^{in}};$$

A wavelet system $\{\psi_{jk}^{(\nu)}\}$ has vanishing moments up to order $n$, $n \in \mathbb{Z}_+$, (**VM$^{\mathbf{n}}$ property** ) if

$$\int\limits_{\mathbb{R}^d} x_1^{\alpha_1}, \dots, x_d^{\alpha_d} \psi^{(\nu)}(x)\, dx = 0, \quad \nu = 1, \dots, r,$$

for all $\alpha \in \mathbb{Z}_+^d$ such that $\|\alpha\|_1 := \sum\limits_{j=1}^{d} \alpha_j \leq n$.

$$\mathbf{VM^n} \iff D^{\alpha}\widehat{\psi}^{(\nu)}(\mathbf{0}) = 0 \quad \forall \|\alpha\|_1 \leq n$$

### Theorem

*Let* $\{\psi_{jk}^{(\nu)}\}$ *be a tight frame. If* $\mathbf{VM^{n-1}}$ *property is fulfilled, then* $\{\psi_{jk}^{(\nu)}\}$ *has approximation order n.*

### Theorem (I.Daubechies, B.Han, A.Ron, Z.Shen, 2003)

*Let* $\{\psi_{jk}^{(\nu)}\}$ *be a tight frame with respect to a diagonal matrix dilation M. If* $\mathbf{VM^{n_0-1}}$ *property is fulfilled and the generating scaling function provides multiresolution approximation order n, then* $\{\psi_{jk}^{(\nu)}\}$ *has approximation order* $\min\{n, 2n_0\}$.

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**Theorem (B.Han, 2003)**

*For arbitrary matrix dilation $M$, there exists a compactly supported tight wavelet frame satisfying* $\mathbf{VM^n}$ *property with an arbitrary $n$.*

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**General scheme for the construction of compactly supported wavelet tight frames – UEP** (A.Ron and Z.Shen, 1997):

$\varphi$ – refinable function, $m_0$ – its mask (a trigonometric polynomial), i.e., $\hat{\varphi}(M^*\xi) = m_0(\xi)\hat{\varphi}(\xi) \quad \forall \xi \in \mathbb{R}^d$.

Let $D(M) = \mathbb{Z}^d / M\mathbb{Z}^d = \{s_0, \dots, s_{m-1}\}$, $s_0 = \mathbf{0}$, (digits), $\mu_{00}, \dots, \mu_{0\,m-1}$ –polyphase components of $m_0$.

$$m_0(x) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i (s_k, x)} \mu_{0k}(M^* x),$$

For example, if $d = 1$, $M = 2$, $D(M) = \{0, 1\}$, then $m_0(\xi) = \frac{1}{\sqrt{2}} \left( \mu_{00}(2\xi) + e^{2\pi i \xi} \mu_{01}(2\xi) \right)$

$$\mathcal{M} = \begin{pmatrix} \mu_{00} & \cdots & \mu_{0\,m-1} \\ \cdots & \cdots & \cdots \\ \mu_{r\,0} & \cdots & \mu_{r\,m-1} \end{pmatrix}, \qquad \mathcal{M}^*\mathcal{M} = I_m.$$

$$m_\nu(x) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i (s_k, x)} \mu_{\nu k}(M^* x) \quad \text{wavelet mask,}$$

$$\widehat{\psi}^{(\nu)}(x) = m_\nu({M^*}^{-1}x)\widehat{\varphi}({M^*}^{-1}x)$$

$\{\psi_{jk}^{(\nu)}\}$ is a tight frame generated by $\varphi$.

Even if $r = m - 1$, the system $\{\psi_{jk}^{(\nu)}\}$ is not necessary an orthonormal basis.

Let $d = 1$, $M = 2$, $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$, $\{\psi_{jk}^{(\nu)}\}$ is generated by $\varphi$

- (a) $m_0(\xi) = (1 + e^{2\pi i\xi})^n t(\xi)$

  It is well known that (a) is necessary and sufficient for $\{\psi_{jk}^{(\nu)}\}$ to have $\mathbf{VM^{n-1}}$ property in the case $r = 1$
- (b) "sum rule"
- (c) polyphase condition:

$$\frac{d^k \mu_{00}}{dx^k}(0) = \frac{\lambda_k}{\sqrt{2}},$$

$$\frac{d^k \mu_{01}}{dx^k}(0) = \frac{1}{\sqrt{2}} \sum_{l=0}^{k} \lambda_l (\pi i)^{k-l}, \quad k = 0, \dots, n-1,$$

$$\lambda_0 = 1, \lambda_l \in \mathbb{C}, l = 1, \dots, n-1.$$

(a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c)

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## Theorem (*Necessary condition for* **VM<sup>n</sup>** *property*)

*If a refinable mask* $m_0$ *generates a tight wavelet frame with* $\mathbf{VM^n}$ *property,* $n > 0$*, then its polyphase components* $\mu_{00}, \dots, \mu_{0\,m-1}$ *satisfy the following conditions*

$$\sum_{k=0}^{m-1} |\mu_{0k}|^2 \le 1 \quad \forall \alpha \in \mathbb{Z}_+^d, \|\alpha\|_1 \le n, \tag{A}$$

$$D^\alpha \mu_{0k}(\mathbf{0}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{0} \le \beta \le \alpha} \lambda_{\alpha-\beta} \prod_{j=1}^{d} \binom{\alpha_j}{\beta_j} \left(-2\pi i (M^{-1} s_k)_j\right)^{\beta_j}, \tag{B}$$

*where* $\lambda_\beta$, $\|\beta\|_1 \le n$, *are complex number such that*

$$\lambda_{\mathbf{0}} = 1, \qquad \sum_{\mathbf{0} \le \beta \le \alpha} \binom{\alpha}{\beta} \lambda_\beta \overline{\lambda_{\alpha-\beta}} = 0, \quad \|\alpha\|_1 \le n \tag{C}$$

## Theorem (*Sufficient condition for* $\mathbf{VM^n}$ *property*)

*If condition* (**B**) *is fulfilled for* $\mu_{00}, \dots, \mu_{0,m-1}$ *and there exists a* $(r+1)\times(r+1)$ *unitary matrix*

$$\begin{pmatrix} \mu_{00} & \cdots & \mu_{0,m-1} & \mu_{0,m} & \cdots & \mu_{0,r} \\ \mu_{10} & \cdots & \mu_{1,m-1} & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{r,0} & \cdots & \mu_{r,m-1} & * & \cdots & * \end{pmatrix}$$

*such that*

$$D^\alpha \mu_{0k}(\mathbf{0}) = 0,\ \ k = m, \dots, r, \alpha \in \mathbb{Z}^d_+, \|\alpha\|_1 \le n, \qquad (\mathbf{D})$$

*then the corresponding wavelet system* $\{\psi^{(\nu)}_{jk}\}$ *is a tight frame with* $VM^n$ *property.*

So, we must extend the row $\mu_{00}, \dots, \mu_{0,m-1}$ such that $1 - \sum_{k=0}^{m-1} |\mu_{0k}|^2 = \sum_{k=m}^{r} |\mu_{0k}|^2$ and (**D**) is fulfilled.

If (**A**) is fulfilled, then $1 - \sum_{k=0}^{m-1} |\mu_{0k}|^2 \geq 0$ **but**

**Open problem:**is it possible to represent an arbitrary non-negative trigonometric polynomials $P$ in the form $P = |f_1|^2 + \dots + |f_N|^2$, where $f_1, \dots, f_N$ are trigonometric polynomials?
Answer YES with $N = 1$ for $d = 1$ (Riesz lemma).

**Theorem** (*M.A.Dritschel, 2005*)

*If $P$ is a strictly positive trigonometric polynomial, then there exist trigonometric polynomials $f_1, \dots, f_N$ so that $P = |f_1|^2 + \dots + |f_N|^2$.*

This does not help directly because $1 - \sum_{k=0}^{m-1} |\mu_{0k}(\mathbf{0})|^2 = 0$.

If the row $\mu_{00}, \dots, \mu_{0,m-1}$ is extended properly, we must extend this row to a unitary matrix whose entries are trigonometric polynomials.
**Open problem:**is it possible to extend any appropriate row of trigonometric polynomials to a unitary matrix whose entries are also trigonometric polynomials?

Answer YES for $d = 1$
(*W. Lawton, S.L. Lee, Z. Shen 1995*)

Answer YES for $2 \times 2$ matrices (evidently)

Conjecture: answer NO in the general setting.

## Construction

**Step 1.** Choose trigonometric polynomials $\mu'_{00}, \dots, \mu'_{0,m-1}$ satisfying **(B)** and **(C)**.

We have already $\sum\limits_{k=0}^{m-1} |\mu'_{0k}(\mathbf{0})|^2 = 1$, but we have no **(A)**:

$\sum\limits_{k=0}^{m-1} |\mu'_{0k}(x)|^2 \le 1 \quad \forall x.$

**Step 2.** Set

$$\mu''_{0k}(x) := \prod_{i=1}^{d} \left(1 - \sin^{2L_i} \pi x_i\right)^{M_i} \mu'_{0k}(x), \ \ k = 0, \dots, m-1,$$

If the numbers $L_j$ and $M_j$ are large enough, we have

$D^\alpha \mu''_{0k}(\mathbf{0}) = D^\alpha \mu'_{0k}(\mathbf{0}), \ \ \|\alpha\|_1 \le n, \ \sum\limits_{k=0}^{m-1} |\mu''_{0k}|^2 \le 3.9.$

**Step 3.** Set $\sigma := \sum_{l=0}^{m-1} |\mu''_{0l}|^2$,

$$\mu_{0k} := \left(\frac{3}{2} - \frac{\sigma}{2}\right) \mu''_{0k}, \quad k = 0, \dots, m-1.$$

We have $D^\alpha \mu_{0k}(\mathbf{0}) = D^\alpha \mu''_{0k}(\mathbf{0})$ $\|\alpha\|_1 \le n$, due to the following

**Theorem**

*If conditions* **(B)** *and* **(C)** *are fulfilled for* $\mu_{00}, \dots, \mu_{0\,m-1}$ *then* $D^\alpha \left(|\mu_{0k}(x)|^2\right)\big|_{x=\mathbf{0}} = 0, \quad k = 0, \dots, m-1,$ *for all* $\alpha \in \mathbb{Z}_+^d$, $0 < \|\alpha\|_1 \le n$, $|\mu_{0k}(\mathbf{0})|^2 = \frac{1}{m}, \quad k = 0, \dots, m-1.$

$$1 - \sum_{k=0}^{m-1} |\mu_{0k}|^2 = 1 - \left(\frac{3}{2} - \frac{\sigma}{2}\right)^2 \sigma = (1-\sigma)^2 \left(1 - \frac{\sigma}{4}\right).$$

Since $1 - \frac{\sigma}{4} \geq \frac{1}{40}$, due to Dritschel's result,

$$1 - \frac{\sigma}{4} = |f_1|^2 + \cdots + |f_N|^2.$$

Set $r = m + N$,

$$\mu_{0m} = (1-\sigma) f_1, \ldots, \mu_{0,r-1} = (1-\sigma) f_N,$$

$$\mu_{0r} = 0.$$

Since $D^\alpha(1-\sigma)(\mathbf{0}) = 0,\ \|\alpha\|_1 \leq n$, $(\mathbf{D})$ is fulfilled.

**Step 4.** Since $\mu_{0r} = 0$, the row $(\mu_{00}, \dots, \mu_{0r})$ may be extended to a unitary matrix

$$\begin{pmatrix} \mu_{00} & \cdots & \mu_{0,m-1} & \mu_{0m} & \cdots & \mu_{0r} \\ \mu_{10} & \cdots & \mu_{1,m-1} & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{m+1,0} & \cdots & \mu_{m+1,m-1} & * & \cdots & * \end{pmatrix}$$

by the Householder formulas

$$\mu_{\nu k} := \delta_{r-\nu,k} - \mu_{0k}\overline{\mu_{0,r-\nu}},$$
$$\mu_{\nu r} := \overline{\mu_{0,r-\nu}}, \quad k = 0, \dots, r, \nu = 1, \dots, r.$$

## Dual frames in a Hilbert space $H$

$\{f_n\} \subset H$ is a frame in $H$ if there exist $A, B > 0$ such that $\forall f \in H$

$$A\|f\|^2 \le \sum_n |\langle f, f_n\rangle|^2 \le B\|f\|^2 \quad \forall f \in H.$$

If $\{f_n\}$ is a frame in $H$, then there exists a dual frame $\{\widetilde{f}_n\}$ in $H$ such that

$$f = \sum_n \langle f, \widetilde{f}_n\rangle f_n \quad \forall f \in H$$

## Dual wavelet frame decomposition

$$f = \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^d} \sum_{\nu=1}^{r} \langle f, \widetilde{\psi}_{jk}^{(\nu)}\rangle \psi_{jk}^{(\nu)} \quad \forall f \in L_2(\mathbb{R}^d)$$

## General scheme for the construction of compactly supported dual wavelet frames (A.Ron and Z.Shen, 1997)

$\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$, $\nu = 1, \dots, r$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$,
$m_0, \widetilde{m}_0$ – refinable masks; $\varphi, \widetilde{\varphi} \in L_2(\mathbb{R}^d)$ – scaling functions,
Polyphase matrices:

$$\mathcal{M} = \begin{pmatrix} \mu_{00} & \cdots & \mu_{0,m-1} \\ \dots & \dots & \dots \\ \mu_{r,0} & \cdots & \mu_{r,m-1} \end{pmatrix} \widetilde{\mathcal{M}} = \begin{pmatrix} \widetilde{\mu}_{00} & \cdots & \widetilde{\mu}_{0,m-1} \\ \dots & \dots & \dots \\ \widetilde{\mu}_{r,0} & \cdots & \widetilde{\mu}_{r,m-1} \end{pmatrix}$$

$$\widetilde{\mathcal{M}}^* \mathcal{M} = I_m,$$

$$\widehat{\psi}^{(\nu)}(x) = m_\nu({M^*}^{-1}x)\widehat{\varphi}({M^*}^{-1}x), \quad \widehat{\widetilde{\psi}}^{(\nu)}(x) = \widetilde{m}_\nu({M^*}^{-1}x)\widehat{\widetilde{\varphi}}({M^*}^{-1}x),$$

### Theorem (B. Han, 2003)

*Let* $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ *be generated by compactly supported refinable functions* $\varphi, \widetilde{\varphi} \in L_2(\mathbb{R}^d)$. *For the systems to be dual frames in* $L_2(\mathbb{R}^d)$ *it is necessary and sufficient they to have* $VM^0$ *property.*

### Theorem

*Let* $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ *be a pair of dual wavelet frames. If* $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ *has* $VM^{n-1}$ *property, then the wavelet frame decomposition has approximation order n.*

We suggest a method for the construction of compactly supported dual wavelet frames with $\mathbf{VM^n}$ property for both the systems $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$

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We suggest a method for the construction of compactly supported dual wavelet frames with $\mathbf{VM^n}$ property for both the systems $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$

**Theorem** (*Necessary condition for* $\mathbf{VM^n}$ *property*)

*If refinable masks* $m_0, \widetilde{m}_0$ *generate compactly supported dual wavelet frames with* $\mathbf{VM^n}$ *property,* $n > 0$*, then their polyphase components* $\mu_{00}, \dots, \mu_{0\,m-1}$, $\widetilde{\mu}_{00}, \dots, \widetilde{\mu}_{0\,m-1}$, *satisfy the following condition: for all* $\alpha \in \mathbb{Z}_+^d$, $\|\alpha\|_1 \le n$,

$$D^\alpha \mu_{0k}(\mathbf{0}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{0}\le\beta\le\alpha} \binom{\alpha}{\beta} \left(-2\pi i (M^{-1}s_k)\right)^\beta \lambda_{\alpha-\beta}, \qquad (\mathbf{B})$$

$$D^\alpha \widetilde{\mu}_{0k}(\mathbf{0}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{0}\le\beta\le\alpha} \binom{\alpha}{\beta} \left(-2\pi i (M^{-1}s_k)\right)^\beta \widetilde{\lambda}_{\alpha-\beta}, \qquad (\widetilde{\mathbf{B}})$$

*where* $\lambda_\beta, \widetilde{\lambda}_\beta$, $\|\beta\|_1 \le n$, *are complex number such that*

$$\lambda_{\mathbf{0}} = \widetilde{\lambda}_{\mathbf{0}} = 1, \qquad \sum_{\mathbf{0}\le\beta\le\alpha} \binom{\alpha}{\beta} \overline{\lambda_\beta} \widetilde{\lambda}_{\alpha-\beta} = 0, \quad \|\alpha\|_1 \le n \qquad (\mathbf{C})$$

## Theorem (*Sufficient condition for* **VM^n** *property*)

*Let* $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ *be MRA-based compactly supported dual wavelet frames, and let*

$$\mathcal{M}^{Ext}=\begin{pmatrix}\mu_{00}&\dots&\mu_{0,m-1}&\mu_{0,m}&\dots&\mu_{0,r}\\ \dots&\dots&\dots&\dots&\dots&\dots\\ \mu_{r,0}&\dots&\mu_{r,m-1}&*&\dots&*\end{pmatrix}$$

$$\widetilde{\mathcal{M}}^{Ext}=\begin{pmatrix}\widetilde{\mu}_{00}&\dots&\widetilde{\mu}_{0,m-1}&\widetilde{\mu}_{0,m}&\dots&\widetilde{\mu}_{0,r}\\ \dots&\dots&\dots&\dots&\dots&\dots\\ \widetilde{\mu}_{r,0}&\dots&\widetilde{\mu}_{r,m-1}&*&\dots&*\end{pmatrix}$$

*be square extensions of their polyphase matrices* $\mathcal{M},\widetilde{\mathcal{M}}$ *such that* $(\widetilde{\mathcal{M}}^{Ext})^*\mathcal{M}^{Ext}=I_{r+1}$. *If* **(B)** *is fulfilled and*

$$D^{\beta}\mu_{0k}(\mathbf{0})=0,\ k=m,\dots,r,\quad\forall\beta\in\mathbb{Z}_+^d,[\beta]\le n,$$

*then* $VM^n$ *property holds for* $\{\widetilde{\psi}_{jk}^{(\nu)}\}$.

## Construction

**Step 1.** Choose an arbitrary refinable function $\varphi$ whose polyphase components $\mu_{00}, \dots, \mu_{0\,m-1}$ satisfy **(B)** with parameters $\lambda_\beta$, $\beta \in \mathbb{Z}_+^d$, $\|\beta\|_1 \le n$, and find dual parameters $\widetilde{\lambda}_\beta$ satisfying **(C)** by the recursive formulas

$$\widetilde{\lambda}_{\mathbf{0}} = 1, \qquad \widetilde{\lambda}_\alpha = -\overline{\lambda_\alpha} - \sum_{\mathbf{0}<\beta\le\alpha} \binom{\alpha}{\beta} \overline{\lambda_\beta}\widetilde{\lambda}_{\alpha-\beta}.$$

**Step 2.** Find trigonometric polynomials $\widetilde{\mu}'_{00}, \dots, \widetilde{\mu}'_{0\,m-1}$ satisfying $(\widetilde{\mathbf{B}})$. Such functions are given by

$$\widetilde{\mu}'_{0\,k}(x) = \frac{1}{\sqrt{m}} \sum_{\|\alpha\|_1 \le n} G_\alpha(x) \sum_{\mathbf{0} \le \beta \le \alpha} \binom{\alpha}{\beta} \left(-2\pi i (M^{-1} s_k)\right)^\beta \widetilde{\lambda}_{\alpha-\beta} + \sum_{[\alpha]=n+1} T_{k,\alpha}(x) \prod_{j=1}^{\alpha_j} \left(1 - e^{2\pi i x_j}\right)^{\alpha_j},$$

where $G_\alpha$ are trigonometric polynomials such that $D^\beta G_\alpha(\mathbf{0}) = \delta_{\alpha\beta}$ for all $\beta \in \mathbb{Z}_+^d$, $\|\beta\|_1 \le n$, $T_{k,\alpha}$ are arbitrary trigonometric polynomials.

**Step 3.** Set $\sigma := \sum\limits_{k=0}^{m-1} \mu_{0k}\, \overline{\widetilde{\mu}'_{0k}}$,
$\widetilde{\mu}_{0k} := (2-\sigma)\widetilde{\mu}'_{0k}, k = 0, \dots, m-1$,
$\mu_{0m} := 1-\sigma, \widetilde{\mu}_{0m} := \overline{1-\sigma}$.

**Theorem**

*If conditions* $(\mathbf{B})$, $(\widetilde{\mathbf{B}})$ *and* $(\mathbf{C})$ *are fulfilled, then* $D^{\beta}\sigma(\mathbf{0}) = 0$ *for all* $\beta \in \mathbb{Z}_{+}^{d}$, $0 < \|\beta\|_1 \le n$.

It follows that $D^{\beta}\widetilde{\mu}_{0k}(\mathbf{0}) = D^{\beta}\widetilde{\mu}'_{0k}(\mathbf{0})\ \|\beta\|_1 \le n$ and
$D^{\beta}\mu_{0m}(\mathbf{0}) = D^{\beta}\widetilde{\mu}'_{0m}(\mathbf{0}) = 0$ for all $\beta \in \mathbb{Z}_{+}^{d}$.
Since $1 - \sum\limits_{k=0}^{m-1} \mu_{0k}(x)\overline{\widetilde{\mu}_{0k}} = (1-\sigma)^2$, we have an appropriate pair of rows $(\mu_{00}, \dots, \mu_{0\,m})$, $(\widetilde{\mu}_{00}, \dots, \widetilde{\mu}_{0\,m})$

**Step 4.** Set $\mu_{0,m+1} \equiv 0, \widetilde{\mu}_{0,m+1} \equiv 0$.
Now we can use the generalized Householder transform for the matrix extension.
For each $\nu = 1, \dots, m+1$, define

$$\widetilde{\mu}_{\nu,m+1} := \overline{\mu_{0,m+1-\nu}}, \quad \mu_{\nu,m+1} := \overline{\widetilde{\mu}_{0,m+1-\nu}},$$
$$\mu_{\nu k} := \delta_{m+1-\nu,k} - \mu_{0k}\overline{\widetilde{\mu}_{0,m+1-\nu}}, \ \widetilde{\mu}_{\nu k} := \delta_{m+1-\nu,k} - \widetilde{\mu}_{0k}\overline{\mu_{0,m+1-\nu}},$$
$$k = 0, \dots, m.$$

Assume that we started with a refinable function $\varphi$ and constructed appropriate polyphase matrices $\mathcal{M}, \widetilde{\mathcal{M}}$. Next we find the masks $m_1, \dots, m_r$, $\widetilde{m}_0, \dots, \widetilde{m}_r$, and define $\widetilde{\varphi}, \psi^{(\nu)}, \widetilde{\psi}^{(\nu)}$, $\nu = 1, \dots, r$, by

$$\widehat{\widetilde{\varphi}}(x) = \prod_{i=1}^{\infty} \widetilde{m}_0({M^*}^{-i} x)$$

$$\widehat{\psi}^{(\nu)}(x) = m_\nu({M^*}^{-1} x)\widehat{\varphi}({M^*}^{-1} x), \quad \widehat{\widetilde{\psi}}^{(\nu)}(x) = \widetilde{m}_\nu({M^*}^{-1} x)\widehat{\widetilde{\varphi}}({M^*}^{-1} x),$$

This construction leads to compactly supported dual wavelet frames $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ if only $\widetilde{\varphi} \in L_2(\mathbb{R}^d)$. **Unfortunately, this is not is not guaranteed**. Generally speaking, $\widetilde{\varphi}$ is a **tempered distribution**.

**What happens if wavelet systems $\{\psi_{\mathbf{jk}}^{(\nu)}\}$, $\{\widetilde{\psi}_{\mathbf{jk}}^{(\nu)}\}$ are constructed according to the general scheme but are not dual wavelet frames in $L_2(\mathbb{R}^d)$ (in particular, if both generating refinable functions, or one of them, are distributions)?**

Let $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ be dual wavelet systems generated by compactly supported refinable functions $\psi, \widetilde{\psi}$, and let $A$ be a class of functions $f$ for which $\langle f, \widetilde{\varphi}_{0k}\rangle$, $\langle f, \widetilde{\psi}_{jk}^{(\nu)}\rangle$ have meaning, We say that $\{\psi_{jk}^{(\nu)}\}$ is frame-like on $A$ if

$$f = \sum_{i=-\infty}^{\infty} \sum_{\nu=1}^{r} \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}_{jk}^{(\nu)}\rangle \psi_{jk}^{(\nu)} \quad \forall f \in A, \tag{1}$$

and $\{\psi_{jk}^{(\nu)}\}$ is almost frame-like on $A$ if

$$f = \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{0k}\rangle \varphi_{0k} + \sum_{i=0}^{\infty} \sum_{\nu=1}^{r} \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}_{jk}^{(\nu)}\rangle \psi_{jk}^{(\nu)} \quad \forall f \in A, \tag{2}$$

where the series in (1) and (2) converge in some natural sense the series over $k$ converge unconditionally.

$S$ is the Schwartz class of functions defined on $\mathbb{R}^d$,
$S'$ is the dual space of $S$

### Theorem

*If* $\varphi, \widetilde{\varphi} \in S'$, *then* $\{\psi_{jk}^{(\nu)}\}$ *is almost frame-like on* $S$ *with the series converging in* $S'$.

### Theorem

*If* $\varphi \in L_2(\mathbb{R}^d)$, $\widetilde{\varphi} \in S'$, *then* $\{\psi_{jk}^{(\nu)}\}$ *is almost frame-like on* $S$ *with the series converging in* $L_2$*-norm.*

### Theorem

*If* $\varphi, \widetilde{\varphi} \in L_2(\mathbb{R}^d)$, *then* $\{\psi_{jk}^{(\nu)}\}$ *is frame-like on* $L_2(\mathbb{R}^d)$ *with the series converging in* $L_2$*-norm.*

Next we assume that the the polyphase components of refinable masks $m_0, \widetilde{m}_0$ satisfy

$$(I) \qquad D^{\beta}\mu_{0k}(\mathbf{0}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{0}\le\gamma\le\beta} \lambda_{\gamma} \binom{\beta}{\gamma} (-2\pi i M^{-1} s_k)^{\beta-\gamma}$$

$$\forall \beta \in \mathbb{Z}_+^d, \|\beta\| < n, k = 0, \dots, m-1;$$

for some complex numbers $\lambda_\gamma$, $\gamma \in \mathbb{Z}_+^d$, $\|\gamma\|_1 < n$, $\lambda_{\mathbf{0}} = 1$, and

$$(II) \quad D^{\beta}\left(1 - \sum_{k=0}^{m-1} \mu_{0k}(\xi)\overline{\widetilde{\mu}_{0k}}(\xi)\right)\Bigg|_{\xi=\mathbf{0}} = 0 \;\; \forall \beta \in \mathbb{Z}_+^d, \|\beta\|_1 < n.$$

## Theorem

*Let* $f \in S$, $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ *be dual wavelet systems generated by compactly supported refinable functions* $\varphi \in L_2(\mathbb{R}^d)$, $\widetilde{\varphi} \in S'$ *satisfying (I), (II). Then*

$$\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\varphi}_{0k} \rangle \varphi_{0k} - \sum_{i=0}^{j-1} \sum_{\nu=1}^{r} \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\psi}_{ik}^{(\nu)} \rangle \psi_{ik}^{(\nu)} \right\|_2 \le \frac{C \|f\|_{W_2^{n^*}}}{(|\lambda| - \varepsilon)^{jn}},$$

*where* $\lambda$ *is a minimal (in module) eigenvalue of* $M$, $\varepsilon > 0$, $|\lambda| - \varepsilon > 1$, $n^* \ge n$, $C$ *and* $n^*$ *do not depend on* $f$ *and* $j$.

## Theorem

*Let* $f \in S$, $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ *be dual wavelet systems generated by compactly supported refinable functions* $\varphi, \widetilde{\varphi} \in L_2(\mathbb{R}^d)$ *satisfying (I), (II). Then for every* $f \in W_2^n(\mathbb{R}^d)$

$$\left\| f - \sum_{i=-\infty}^{j-1} \sum_{\nu=1}^{r} \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\psi}_{ik}^{(\nu)} \rangle \psi_{ik}^{(\nu)} \right\|_2 \le \frac{C \|f\|_{W_2^n}}{(|\lambda| - \varepsilon)^{jn}},$$

*where* $\lambda$ *is a minimal (in module) eigenvalue of* $M$, $\varepsilon > 0$, $|\lambda| - \varepsilon > 1$, $C$ *does not depend on* $f$ *and* $j$.

$$L^0_\infty(\mathbb{R}^d) := \left\{ f \in L_\infty(\mathbb{R}^d) : \ \operatorname*{esssup}_{|x|\ge R} |f(x)| \xrightarrow[R\to\infty]{} 0 \right\}$$

**Theorem**

*Let* $\{\psi^{(\nu)}_{jk}\}$, $\{\widetilde{\psi}^{(\nu)}_{jk}\}$ *be dual wavelet systems generated by compactly supported refinable functions* $\varphi \in L_\infty(\mathbb{R}^d)$, $\widetilde{\varphi} \in L(\mathbb{R}^d)$, *satisfying (I), (II). Then for every* $f \in W^n_\infty \cap L^0_\infty(\mathbb{R}^d)$

$$\left\| f - \sum_{i=-\infty}^{j-1} \sum_{\nu=1}^{r} \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}^{(\nu)}_{ik}\rangle \psi^{(\nu)}_{ik} \right\|_\infty \le \frac{C\|f\|_{W^n_\infty}}{(|\lambda|-\varepsilon)^{jn}},$$

*where* $\lambda$ *is a minimal (in module) eigenvalue of* $M$, $\varepsilon > 0$, $|\lambda| - \varepsilon > 1$, $n^* \ge n$, $C$ *and* $n^*$ *do not depend on* $f$ *and* $j$.

### Theorem

*Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ *be dual wavelet systems generated by compactly supported refinable functions* $\varphi \in L_\infty(\mathbb{R}^d)$, $\widetilde{\varphi} \in L_q(\mathbb{R}^d)$, *satisfying (I), (II). Then for every* $f \in W_p^n$

$$\left\| f - \sum_{i=-\infty}^{j-1} \sum_{\nu=1}^{r} \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\psi}_{ik}^{(\nu)} \rangle \psi_{ik}^{(\nu)} \right\|_p \le \frac{C \|f\|_{W_p^n}}{(|\lambda| - \varepsilon)^{jn}},$$

*where* $\lambda$ *is a minimal (in module) eigenvalue of* $M$, $\varepsilon > 0$, $|\lambda| - \varepsilon > 1$, $C$ *does not depend on* $f$ *and* $j$.

## Theorem

*Let* $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ *be dual wavelet systems generated by compactly supported refinable functions* $\varphi, \widetilde{\varphi} \in L_\infty(\mathbb{R}^d)$ *satisfying (I), (II). Then for every* $f \in W_1^n$

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{0k}\rangle \varphi_{0k} - \sum_{i=0}^{j-1} \sum_{\nu=1}^{r} \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}_{ik}^{(\nu)}\rangle \psi_{ik}^{(\nu)} \right\|_1 \le \frac{C\|f\|_{W_1^n}}{(|\lambda|-\varepsilon)^{jn}},$$

*where* $\lambda$ *is a minimal (in module) eigenvalue of* $M$, $\varepsilon > 0$, $|\lambda| - \varepsilon > 1$, $C$ *does not depend on* $f$ *and* $j$.

## Examples

1. $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ , $m = 2,\ D(M) = \{s_0 = (0,0), s_1 = (1,0)\}$

$$\widehat{\varphi}(\xi) = \left(\frac{1 - e^{-2\pi i\xi_1}}{2\pi i\xi_1}\right)\left(\frac{1 - e^{-2\pi i\xi_2}}{2\pi i\xi_2}\right)^2\left(\frac{1 - e^{-2\pi i(\xi_1+\xi_2)}}{2\pi i(\xi_1+\xi_2)}\right)\left(\frac{1 - e^{-2\pi i(\xi_1-\xi_2)}}{2\pi i(\xi_1-\xi_2)}\right)$$

$\varphi \in C^2$,

a) $\widetilde{\varphi}$ is the $\delta$-function, $\widetilde{\varphi} \in S'$

$\psi^{(1)}(x) = \sqrt{2}\varphi(Mx + \begin{pmatrix} 1 \\ 0 \end{pmatrix})$, $\psi^{(2)}(x) = \sqrt{2}\varphi(Mx)$ are in $C^2$ and symmetric with respect to the points $(-\frac{7}{4}, -\frac{5}{4})$, $(-\frac{5}{4}, -\frac{3}{4})$ respectively

$$f(x)=\sum_{k\in\mathbb{Z}^2}f(-k)\varphi(x+k)+\sum_{j=0}^{+\infty}\sum_{k\in\mathbb{Z}^2}\left[-\frac{1}{2}f\left(-M^{-j-1}\left(Mk+\begin{pmatrix}1\\1\end{pmatrix}\right)\right)-\right.$$

$$\frac{1}{4}f\left(-M^{-j-1}\left(Mk+\begin{pmatrix}2\\2\end{pmatrix}\right)\right)-\frac{1}{4}f\left(-M^{-j-1}\left(Mk+\begin{pmatrix}2\\0\end{pmatrix}\right)\right)+$$

$$\left.f\left(-M^{-j-1}\left(Mk+\begin{pmatrix}1\\0\end{pmatrix}\right)\right)\right]\varphi\left(M^{j+1}x+Mk+\begin{pmatrix}1\\0\end{pmatrix}\right)+$$

$$\sum_{j=0}^{+\infty}\sum_{k\in\mathbb{Z}^2}\left[\frac{3}{4}f\left(-M^{-j}k\right)-\frac{1}{4}f\left(-M^{-j-1}\left(Mk+\begin{pmatrix}0\\2\end{pmatrix}\right)\right)-\right.$$

$$\left.\frac{1}{2}f\left(-M^{-j-1}\left(Mk+\begin{pmatrix}1\\1\end{pmatrix}\right)\right)\right]\varphi(M^{j+1}x+Mk).$$

This expansion holds for every $f\in S$ and has approximation order 1.

b) To provide approximation order 2 for the same refinable function $\varphi$ we can take

$$\widetilde{m}_0(\xi) = 1-\frac{3}{8}e^{2\pi i(\xi_1+\xi_2)}+\frac{3}{8}e^{-2\pi i(\xi_1+\xi_2)}+\frac{1}{8}e^{2\pi i(\xi_1-\xi_2)}-\frac{1}{8}e^{-2\pi i(\xi_1-\xi_2)}.$$

The dual wavelet masks $\widetilde{m}_1$, $\widetilde{m}_2$ are given by

$$\frac{1}{32\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -18 & 0 & 2 \\ 0 & 0 & 0 & -6 & 32 & -10 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{1}{32\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & -6 & 0 & 6 \\ 0 & -2 & 0 & -14 & 0 \\ 0 & 0 & 18 & 0 & -2 \\ 0 & -3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2. $d = 1$, $M = 2$, $\varphi$ and $\widetilde{\varphi}$ are B-splines:

$$\widehat{\varphi}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^4, \quad \widehat{\widetilde{\varphi}}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2.$$

$$\begin{aligned}
\psi^{(1)}(x) &= \sqrt{2}\varphi(2x+1), \\
\psi^{(2)}(x) &= \sqrt{2}\varphi(2x), \\
\widetilde{\psi}^{(1)}(x) &= \frac{-1}{2\sqrt{2}}\left(\frac{1}{2}\widetilde{\varphi}(2x-1) + \widetilde{\varphi}(2x) - 3\widetilde{\varphi}(2x+1) + \widetilde{\varphi}(2x+2) + \frac{1}{2}\widetilde{\varphi}(2x+3)\right) \\
\widetilde{\psi}^{(2)}(x) &= \frac{1}{8\sqrt{2}}\left(-\frac{1}{2}\widetilde{\varphi}(2x-3) - \widetilde{\varphi}(2x-2) - \frac{7}{2}\widetilde{\varphi}(2x-1) + 10\widetilde{\varphi}(2x) - \right. \\
&\qquad \left. \frac{7}{2}\widetilde{\varphi}(2x+1) - \widetilde{\varphi}(2x+2) - \frac{1}{2}\widetilde{\varphi}(2x+3)\right).
\end{aligned}$$

$\{\psi_{jk}^{(\nu)}\}$ is frame-like with approximation order 2 on $W_p^2(\mathbb{R})$ if $1 < p < \infty$, on $W_\infty^2(\mathbb{R}) \cap L_\infty^0(\mathbb{R})$ if $p = \infty$, and almost frame-like with approximation order 2 on $W_1^2(\mathbb{R})$.