IMAHA Open Problems Discussion Session Notes NIU, April 24, 2010

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Summary

The following slides summarize discussion of a number of open problems in applied harmonic analysis that were considered during the “open problems” session at the end of the IMAHA 2010 meeting. A number of technical side problems were also discussed during the five main lectures presented throughout the meeting. Chris Heil’s lecture in particular was dedicated to open problems, and his lecture slides list 15 such problems. The discussion followed the topics of the lectures, and included problems associated with ...

- representations by time-frequency shifts
- signal recovery from transform magnitudes
- fractional or generalized Fourier transforms
- function representations via transformation groups
- approximation order for wavelet-like systems

Specific problems that were suggested

Shannon wavelets and L^p

The Shannon wavelet basis has the form $\{2^{j/2}S(2^j \cdot -k)\}_{j,k\in\mathbb{Z}}$ where S is the function whose Fourier transform is the characteristic function of $[-1,-1/2]\cup[1/2,1]$. Does the Shannon wavelet basis span L^p where $1 < p < 2$? [suggested by Richard Laugesen]

A discrete criterion for M^1?

Let $g_{a,b}(t) = \mathrm{e}^{2\pi \mathrm{i} at}\, g(t-b)$. Given a lattice Gabor system $\mathcal{G}(g,\alpha,\beta) = \{g_{m\alpha,n\beta}\}_{m,n\in\mathbb{Z}}$, $\alpha\beta < 1$ if $\sum |\langle g, g_{m\alpha,n\beta}\rangle| < \infty$ and $\mathcal{G}(g,\alpha,\beta)$ forms a frame for L^2, then is $g \in M^1$, the space of functions whose short-time Fourier transforms belong to $L^1(\mathbb{R}^2)$? [suggested by Radu Balan]

Closure of M^1 in "frame norm"?

What is the closure of M^1 with respect to the square root of the norm of the Gabor frame operator or, equivalently, the norm of the coefficient mapping $f \mapsto \{\langle f,\, f_{m\alpha,n\beta}\rangle\}$? What are the corresponding closures of the Wiener spaces $W(C_0, \ell^1)$ and $W(C_0, \ell^1) \cap \widehat{W(C_0, \ell^1)}$? [suggested by Radu Balan]

Time-frequency localized bases?

Does there exist a Riesz basis $\{g_k\} \subset L^2(\mathbb{R})$ and a discrete set $\Lambda = \{\lambda_k\} \subset \mathbb{R}^2$ such that $|\mathrm{STFT}(g_k, \gamma)(\cdot)| \leq F(\cdot - \lambda_k)$ for some fixed $F \in L^1(\mathbb{R}^2)$? Here γ is the Gaussian window. (cf. Bourgain's work. Here, $\{g_k\}$ could be time-frequency shifts of a fixed function or a more general collection). [suggested by Radu Balan]

Balian-Low for aggregate systems?

Is it possible to circumvent the Balian-Low problem by using different Gabor functions for different subsets of time-frequency shifts? Let S and T partition $\mathbb{Z}^2$, that is $S \cup T = \mathbb{Z}^2$ and $S \cap T = \emptyset$. Do there exist functions g and h in $L^2(\mathbb{R})$ satisfying $\int |t\, g(t)|^2 \, \mathrm{d}t < \infty$, $\int |t\, h(t)|^2 \, \mathrm{d}t < \infty$, $\int |\xi\, \widehat{g}(\xi)|^2 \, \mathrm{d}\xi < \infty$ and $\int |\xi\, \widehat{h}(\xi)|^2 \, \mathrm{d}\xi < \infty$ such that $\{g_{m,n}(t) : (m,n) \in S\} \bigcup \{h_{m,n}(t) : (m,n) \in T\}$ forms a Riesz basis or frame etc. for $L^2(\mathbb{R})$? Here $g_{m,n}(t) = \mathrm{e}^{2\pi \mathrm{i} m t} g(t-n)$. This is a distinct problem from characterizing multiple-window Gabor systems such as in Zibulski and Zeevi's work. A similar problem could be phrased for exact systems with corresponding "Balian-Low" mixed Sobolev conditions. [suggested by Chris Heil]

It was proved by Šikić and Nielsen that the shifts $\varphi(\cdot - k)$, $k \in \mathbb{Z}$, of $\varphi \in L^2(\mathbb{R})$ form a Schauder basis for their span if and only if the periodization $w = \sum |\widehat{\varphi}(\xi + k)|^2$ is an A_2-weight. One the other hand, it is known that every A_p-weight is in $A_{p-\epsilon}$ for some $\epsilon > 0$. Is this improvement reflected in an additional property of the exponential Schauder basis for $L^2(w)$, the space of periodic functions square integrable with respect to w on $[0, 1]$? [suggested by Joe Lakey]

Applications of wavelet-like systems?

A number of applications have been suggested for shearlets, particularly imaging applications that might take advantage of their directional sensitivity. What are the (known or potential) applications for hyperbolic wavelet systems and other types of (composite) wavelet systems? [suggested by Ed Wilson]