A Selection of Problems in Time-Frequency Analysis and Wavelet Theory

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CHALLENGE FOR THE HOUR

Problem 1: HRT Subconjecture

If $g \in L^2(\mathbb{R})$ is nonzero, then

$$\left\{ g(x), \quad g(x-1), \quad e^{2\pi i x} g(x), \quad e^{2\pi i \sqrt{2} x} g(x-\sqrt{2}) \right\}$$

is a linearly independent set of functions in $L^2(\mathbb{R})$.

<u>HRT Subconjecture</u>

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<u>The HRT Subsubconjecture</u>

If $g \in \mathcal{S}(\mathbb{R})$ is nonzero, then

$$\left\{g(x), \quad g(x-1), \quad e^{2\pi i x}g(x), \quad e^{2\pi i\sqrt{2}x}g(x-\sqrt{2})\right\}$$

is a linearly independent set of functions in $L^2(\mathbb{R})$.

TRANSLATIONS

Theorem

Given $g \in L^2(\mathbb{R})$, **define** $\Phi_g(\xi) \;=\; \sum_{k\in\mathbb{Z}} |\widehat{g}(\xi+k)|^2$. **Then:**

(a) $\{T_k g\}_{g\in\mathbb{Z}}$ **is Bessel** $\iff \Phi_g \in L^\infty(\mathbb{T})$.

(b) $\{T_k g\}_{g\in\mathbb{Z}}$ **is minimal** $\iff 1/\Phi_g \in L^1(\mathbb{T})$.

(c) $\{T_k g\}_{g\in\mathbb{Z}}$ **is a frame for** $V_0 \iff A \le \Phi_g(\xi) \le B$ **a.e. off** Z_{Φ_g}.

(d) $\{T_k g\}_{g\in\mathbb{Z}}$ **is an unconditional basis for** $V_0 \iff A \le \Phi_g \le B$ **a.e.**

(e) $\{T_k g\}_{g\in\mathbb{Z}}$ **is an orthonormal basis for** $V_0 \iff \Phi_g(t) = 1$ **a.e.**

(f) With respect to the ordering $\{0, -1, 1, -2, 2, \dots\}$,

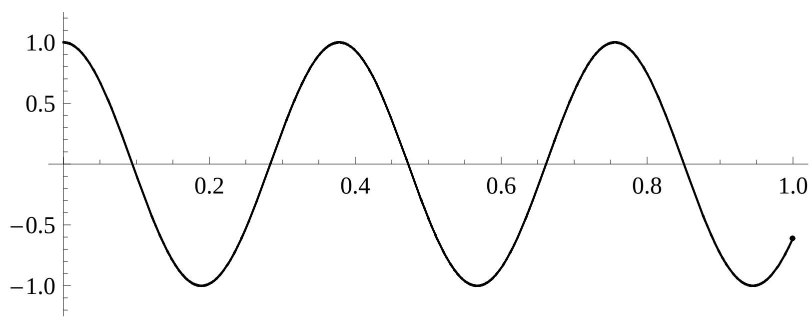
$\{T_k g\}_{g\in\mathbb{Z}}$ **is a Schauder basis for** $V_0 \iff \Phi_g \in \mathcal{A}_2(\mathbb{T})$.

Irregular Translations

Translations are always finitely independent in $L^2(\mathbb{R})$:

$$\sum_{k=1}^{N} c_k g(x-a_k) = 0 \quad \Longrightarrow \quad \left(\sum_{k=1}^{N} c_k e^{2\pi i a_k \xi}\right) \widehat{g}(\xi) = 0 \quad \Longrightarrow \quad \widehat{g} = 0.$$

What if $p \neq 2$?



Translates are dependent in $L^\infty(\mathbb{R})$.

Consider $2 < p < \infty$:

$$\sum_{k=1}^{N} c_k g(x - a_k) = 0 \quad \Longrightarrow \quad \varphi \, \widehat{g} = 0 \quad \Longrightarrow \quad ?$$

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$$g = 1 \quad \Longrightarrow \quad g(x) - g(x-1) = 0 \quad \Longrightarrow \quad (e^{2\pi i \xi} - 1)\,\widehat{g} = 0.$$

Consider $2 < p < \infty$:

$$\sum_{k=1}^{N} c_k g(x - a_k) = 0 \qquad \Longrightarrow \qquad \varphi\,\widehat{g} = 0 \qquad \Longrightarrow \qquad ?$$

$$g = 1 \qquad \Longrightarrow \qquad g(x) - g(x-1) = 0 \qquad \Longrightarrow \qquad (e^{2\pi i\xi} - 1)\,\widehat{g} = 0.$$

$$\varphi\,\delta = 0 \qquad \not\Longrightarrow \qquad \varphi = 0.$$

Distributions can be supported at a single point, functions cannot.

$$\textbf{Q.}\ \exists\, g \in L^p(\mathbb{R}) \quad \textbf{s.t.} \quad \left(\sum_{k=1}^{N} c_k e^{2\pi i \xi a_k}\right) \widehat{g}(\xi) = 0?$$

<u>**Theorem**</u> **(Edgar/Rosenblatt, 1979)**

Assume $S \subseteq \mathbb{R}^n$ **is closed and has Hausdorff dimension** $\leq n-1$. **If** $g \in L^p(\mathbb{R}^n)$ **and** $\operatorname{supp}(\widehat{g}) \subseteq S$, **then** $p \geq 2n/(n-1)$.

<u>**Corollary**</u>

If $g \in L^p(\mathbb{R}^n)$ **and** $p < 2n/(n-1)$, **then translates of** g **are linearly independent.**

<u>**Proof sketch**</u>

The zero set of a trigonometric polynomial φ **on** $\mathbb{R}^n$ **is the intersection of an analytic variety in** $\mathbb{C}^n$ **with** $\mathbb{R}^n$. **The Hausdorff dimension of this zero set is** $\leq n-1$.

Corollary

If $g \in L^p(\mathbb{R}^n)$ **and** $p < 2n/(n-1)$, **then translates of** g **are linearly independent.**

Theorem (Edgar/Rosenblatt, 1979)

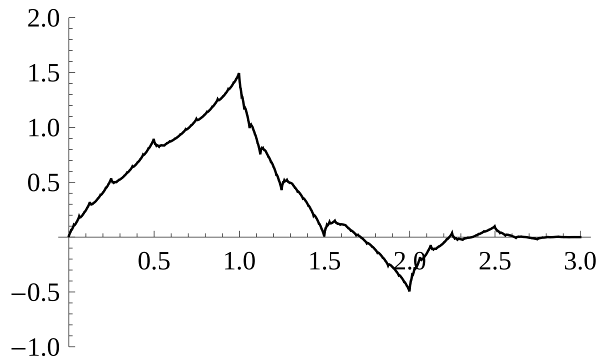
If $2n/(n-1) < p < \infty$, **then there exists a function** $g \in L^p(\mathbb{R}^n)$ **that has linearly dependent translates.**

Theorem (Rosenblatt, 1995)

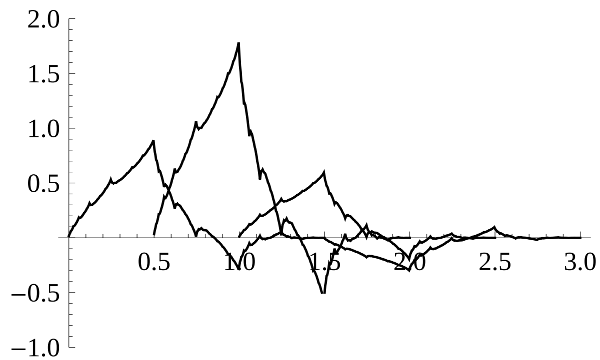
If $g \in L^p(\mathbb{R}^n)$ **and** $p = 2n/(n-1)$, **then translates of** g **are linearly independent.**

Thus, translates are independent in $L^1(\mathbb{R})$ **for** $1 \le p < \infty$, **but translates are independent in** $L^2(\mathbb{R}^2)$ **for** $1 \le p \le 4$.

Time-scale translates can be linearly dependent:



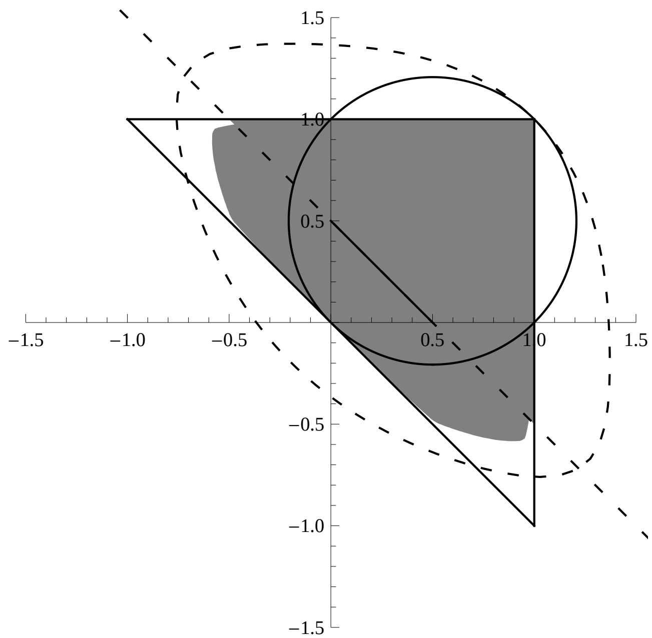
A scaling function



$$\varphi(x) = \tfrac{3}{5}\varphi(2x) + \tfrac{6}{5}\varphi(2x-1) + \tfrac{2}{5}\varphi(2x-2) - \tfrac{1}{5}\varphi(2x-3)$$

Wavelet Path-Connectedness

`http://cm.bell-labs.com/who/wim/cascade/`



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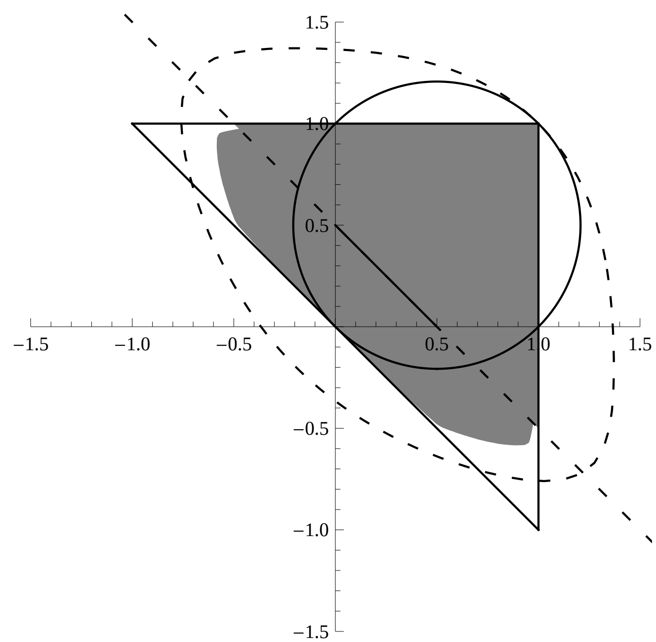

Problem 2:

Are the continuous wavelets path-connected in the uniform norm?

Problem 3: HRT Conjecture (H/Ramanathan/Topiwala, 1996)

If $g \in L^2(\mathbb{R}) \setminus \{0\}$ **and** $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subseteq \mathbb{R}^2$ **is finite, then**

$$\mathcal{G}(g, \Lambda) = \{e^{2\pi i b_k x} g(x - a_k)\}_{k=1}^N$$

is linearly independent.

Known for: g **compactly supported,**

$|\Lambda| \leq 3,$

$\Lambda \subseteq$ **lattice (Linnell; new proof Bownik/Speegle).**

Unknown for: $g \in \mathcal{S}(\mathbb{R})$**,**

$|\Lambda| \geq 4$**.**

Perturbation theorem:

$$\mathcal{G}(g, \Lambda) \text{ independent} \quad \Longrightarrow \quad \text{can perturb } g \text{ and } \Lambda.$$

Proof

Write $\Lambda = \{(a_k, b_k)\}_{k=1}^N$**.**

If $\mathcal{G}(g, \Lambda)$ **is independent then it is a Riesz basis for its span:**

$$A \sum_{k=1}^N |c_k| \;\leq\; \left\| \sum_{k=1}^N c_k \, M_{b_k} T_{a_k} g \right\| \;\leq\; B \sum_{k=1}^N |c_k|.$$

If $\|g - h\| < A$**, then**

$$\begin{aligned} \left\| \sum_{k=1}^N c_k \, M_{b_k} T_{a_k} h \right\|_2 \;&\geq\; \left\| \sum_{k=1}^N c_k \, M_{b_k} T_{a_k} g \right\|_2 \;-\; \left\| \sum_{k=1}^N c_k \, M_{b_k} T_{a_k} (h-g) \right\|_2 \\ &\geq\; A \sum_{k=1}^N |c_k| \;-\; \sum_{k=1}^N |c_k| \, \|M_{b_k} T_{a_k} (g-h)\|_2 \\ &=\; \big(A - \|g-h\|_2\big) \; \sum_{k=1}^N |c_k|. \end{aligned}$$

Problem 4: Explicit frame bounds

Given $g \in L^2(\mathbb{R})$ and a finite set $\Lambda \subset \mathbb{R}^2$, find explicit values for the frame bounds of a Gabor system $\mathcal{G}(g, \Lambda)$ or a wavelet system $\mathcal{W}(g, \Lambda)$ as a frame for its span in $L^2(\mathbb{R})$.

Sufficiently "explicit" control of the frame bounds of a Gabor system would settle the HRT Conjecture.

Theorem [**Christensen and Lindner, 2001**]

If $\delta = \inf_{j \neq k} |\lambda_j - \lambda_k| > 0$ then $\{e^{i\lambda_k x}\}_{k=1}^{N}$ is a Riesz basis for its span in $L^2(-\pi, \pi)$ with lower frame bound

$$A_N \;=\; 1.6 \times 10^{-14}\, (\delta/2)^{2N+1}\, ((N+1)!)^{-8}.$$

Problem 5: Do better! For exponentials, Gabor, wavelets, and more.

Problem 6: Connect to the frame potential

$$\mathbf{FP}(\{x_n\}_{n=1}^N) = \sum_{m=1}^N \sum_{n=1}^N |\langle x_m, x_n\rangle|^2.$$

Local minimizer of the frame potential $\iff$ FUNTF.

Problem 7: Perturbation theory for the frame potential. How does the frame potential of a finite Gabor system $\{e^{2\pi i b_k x} g(x - a_k)\}_{k=1}^N$ vary as we perturb g or (a_k, b_k)?

Problem 8: Shape of the minima. Is FP similar around any local minimizer? Are some local minimizers better than others?

BASES OF TRANSLATES

Set

$$V_0 \;=\; \overline{\mathrm{span}}\big\{g(x-a_k)\big\}_{k\in\mathbb{N}}.$$

Q. How large can it be?

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Q. How large can it be?

<u>Theorem</u> (Olevskii, 1997; Olevskii/Ulanovskii, 2004)

There exist nice $g \in L^2(\mathbb{R})$ and $|a_k - k| < \varepsilon$ such that $V_0 = L^2(\mathbb{R})$.

Q. How nice can $\{g(x-a_k)\}_{k\in\mathbb{N}}$ be?

<u>Theorem</u> (Olson/Zalik, 1992)

It can't be a Riesz basis for $L^2(\mathbb{R})$.

Olson/Zalik Conjecture: It can't be a Schauder basis for $L^2(\mathbb{R})$.

Theorem (Christensen/Deng/H., 1999)

It can't be a frame for $L^2(\mathbb{R})$.

Theorem (Deng/H., 2000)

If $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, **then it can't be a Schauder basis for** $L^2(\mathbb{R})$.

Proof Sketch

A Homogeneous Approximation Property is satisfied.

If $\{T_a g\}_{a\in\Gamma}$ is a basis, then:

$$\begin{aligned}
\Big\|\chi - \sum_{a\in[-R,R]} \langle \chi, T_a g\rangle\, \widetilde{g}_k\Big\|_2 &= \Big\|\sum_{a\notin[-R,R]} \langle \chi, T_a g\rangle\, \widetilde{g}_a\Big\|_2 \\
&\le \sum_{a\notin[-R,R]} |\langle \chi, T_a g\rangle|\, \|\widetilde{g}_a\|_2 \\
&\le C \sum_{a\notin[-R+1,R-1]} \int_a^{a+1} |g| \\
&\le C \int_{|x|>R-1} |g| \\
&\le \varepsilon.
\end{aligned}$$

Similarly,

$$\left\| M_\eta T_u \chi \; - \sum_{a\in[-R+u,R+u]} \langle \chi, T_a g\rangle \, \widetilde{g}_k \right\|_2 \; \le \; \varepsilon.$$

A Ramanathan/Steger double projection argument then implies a "Comparison Theorem":

$$D^+(\mathbb{Z}^2) \;\le\; D^+(\Gamma \times \{0\}),$$

which is a contradiction.

Problem 9: Does $\exists\, g \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$ such that $\{g(x-a_k)\}_{k\in\Gamma}$ is a Schauder basis for $L^2(\mathbb{R})$?

NYQUIST DENSITY

Feichtinger Algebra: $g \in M^1 \iff \iint |V_\varphi g| < \infty \approx f, \widehat{f} \in L^1.$

STFT: $V_\varphi g(x, \xi) = \langle g, M_\xi T_x g \rangle.$

HAP $\implies$ **density** $\geq$ **density of a R.B. or S.B.**

<u>**Theorem**</u>

$$g \in L^2, \textbf{Frame} \implies \textbf{HAP} \implies D^-(\Lambda) \geq 1,$$

$$g \in L^2, \textbf{Schauder basis} \implies (\textit{nothing}) \implies D^+(\Lambda) \leq 1,$$

$$g \in L^2, \textbf{Riesz basis} \implies \textbf{HAP} \implies D^\pm(\Lambda) = 1,$$

$$g \in M^1, \textbf{Schauder basis} \implies \textbf{HAP} \implies D^\pm(\Lambda) = 1,$$

Problem 10: **H/Deng Conjecture:**

$$D^{\pm}(\Lambda) = 1 \textbf{ for all Gabor Schauder bases } G(g, \Lambda).$$

Lattice Gabor Systems

$$g \in L^2, \textbf{complete}, \Lambda \textbf{ lattice} \implies \textbf{HAP} \implies D^-(\Lambda) \geq 1,$$

$$g \in L^2, \textbf{Schauder basis} \implies (\textit{nothing}) \implies D^+(\Lambda) \leq 1,$$

$$g \in L^2, \textbf{Schauder basis}, \Lambda \textbf{ lattice} \implies \textbf{HAP} \implies D^-(\Lambda) \geq 1,$$

Theorem **(H/Powell, 2006)**

$$G(g, \mathbb{Z}^2) \textbf{ is a Schauder basis} \iff |Zg|^2 \in \mathcal{A}(\mathbb{T} \times \mathbb{T}).$$

Affine Wavelet System

$$\left\{a^{n/2}\psi(a^n x - bk)\right\}_{k,n\in\mathbb{Z}} \;=\; \{D_{a^n} T_{bk}\psi\}_{k,n\in\mathbb{Z}}$$

The density of $\Lambda = \{(a^n, bk)\}$ in the affine group $\mathbb{A} = \mathbb{R}^+ \times \mathbb{R}$ is

$$D^{\pm}(\Lambda) \;=\; \frac{1}{b \ln a}.$$

Co-Affine Wavelet System

$$\left\{a^{n/2}\psi(a^n x - bk)\right\}_{k,n\in\mathbb{Z}} \;=\; \{T_{bk} D_{a^n}\psi\}_{k,n\in\mathbb{Z}} \;=\; \{D_{a^n} T_{a^{-n}bk}\psi\}_{k,n\in\mathbb{Z}}$$

Theorem (Gressman, Labate, Weiss, and Wilson, 2003)

No co-affine wavelet system can be a frame (even weighted).

Co-Affine Density

The density of $\Lambda = \{(a^n, a^{-n}bk)\}$ in the affine group is

$$D^{-}(\Lambda) \;=\; 0, \qquad D^{+}(\Lambda) \;=\; \infty.$$

<u>No Nyquist Density for Wavelets</u>

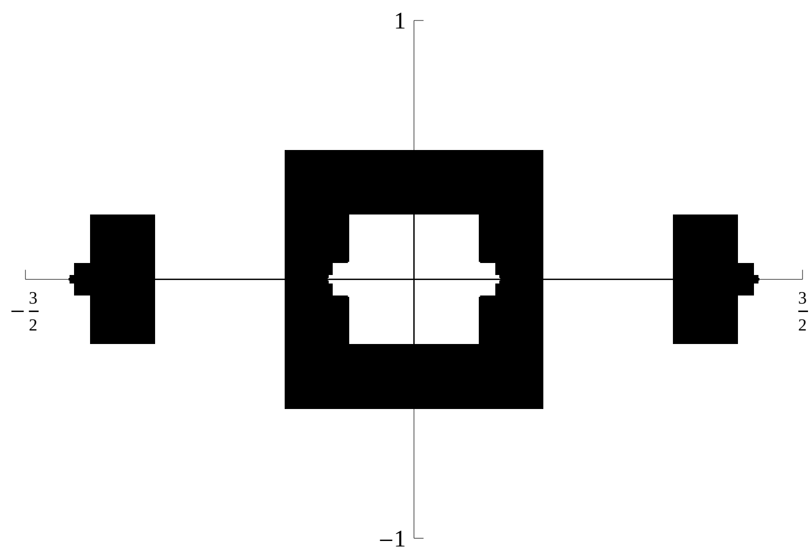
Given any $a > 1$, $b > 0$, there exists $\psi \in L^2(\mathbb{R})$ such that

$$\left\{a^{n/2}\psi(a^n x - bk)\right\}_{n,k\in\mathbb{Z}}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

<u>The wedding cake wavelet set</u> (Dai, Larson, Speegle)

$A = 2I$**,** $b = 1$



<u>Theorem</u> (H/Kutyniok, 2007)

A HAP holds for irregular wavelet frames $\left\{a^{1/2}\psi(ax-b)\right\}_{(a,b)\in\Lambda}$ **generated by** $\psi \in L^2(\mathbb{R})$ **with**

(b) $\psi \in C^1(\mathbb{R})$,

(a) $|\psi(x)| \leq C(1+|x|)^{-\alpha}$ **for some** $C > 0$ **and** $\alpha > 2$,

(c) $\hat{\psi}(0) = 0$.

Yet

$$\textbf{HAP} \quad \not\Rightarrow \quad \textbf{Nyquist Density!}$$

<u>Problem 11:</u>

Why? Characterize those groups and representations for which HAP holds and implies a Nyquist density.

Theorem (H/Kutyniok, 2003, 2007)

(a) Wavelet frame $\implies D^+(\Lambda) < \infty$.

(b) Wavelet frame + HAP $\implies 0 < D^-(\Lambda)$, **for** ψ **in our special class.**

Problem 12:

Does $\exists\, \psi \in L^2(\mathbb{R})$ **and** $D^-(\Lambda) = 0$ **such that** $\left\{a^{1/2}\psi(ax-b)\right\}_{(a,b)\in\Lambda}$ **is a frame for** $L^2(\mathbb{R})$**?**

LATTICE GABOR FRAMES

Theorem (Lyubarskii and Seip/Wallstén, 1992)

For $\varphi(t) = 2^{1/4} e^{-\pi t^2}$,

$$\mathcal{G}(\varphi, \Lambda) \text{ frame } \iff D^-(\Lambda) > 1 \text{ (+ separation criteria)}.$$

Problem 13: **Gröchenig/Lyubarskii Conjecture**

For $\varphi_N = N$**th Hermite Function,**

$$\mathcal{G}(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z}) \text{ frame } \iff D^-(\Lambda) > N.$$

Theorem (Balan/Casazza/H/Landau, 2006)

Frame + $D^-(\Lambda) > 1 \implies \exists\, D^{\pm}(\Lambda') < \varepsilon$ **s.t.** $\mathcal{G}(g, \Lambda \setminus \Lambda')$ **is a frame.**

Theorem (Balan/Casazza/Landau, 2009)

Frame + $D^-(\Lambda) > 1 \implies \exists\, 1 \leq D^{\pm}(\Lambda') < 1 + \varepsilon$ **s.t.** $\mathcal{G}(g, \Lambda')$ **is a frame.**

Theorem (Bekka, 2004)

If Λ is a lattice in $\mathbb{R}^{2d}$ then $\exists\, g \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(g,\Lambda)$ is a frame.

Example

Gaussian generator: $\phi(x_1,x_2) = e^{-2\pi i(x_1^2+x_2^2)}$

Rectangular lattice: $\Lambda = \left(2\mathbb{Z} \times \frac{1}{4}\mathbb{Z}\right) \times \left(2\mathbb{Z} \times \frac{1}{4}\mathbb{Z}\right)$

$\left\{M_{(2n_1,n_2/4)}T_{2k_1,2n_2/4}\phi\right\}_{k,n\in\mathbb{Z}^2}$ **is incomplete in** $L^2(\mathbb{R}^2)$.

Theorem (Bekka, 2004)

If Λ is a lattice in $\mathbb{R}^{2d}$ then $\exists\, g \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(g,\Lambda)$ is a frame.

Note: Proof is nonconstructive.

Constructive proofs exist for symplectic lattices (easy) and separable lattices (Han/Wang, 2001).

$$\left(2\mathbb{Z} \times \frac{1}{2}\mathbb{Z}\right) \times \left(2\mathbb{Z} \times \frac{1}{2}\mathbb{Z}\right)$$

is nonsymplectic.

Problem 14: Do there always exist compactly supported g such that $\mathcal{G}(g,\Lambda)$ is a frame?

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is a linearly independent set of functions in $L^2(\mathbb{R})$.

The HRT Subsubconjecture

If $g \in \mathcal{S}(\mathbb{R})$ is nonzero, then

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THANK YOU

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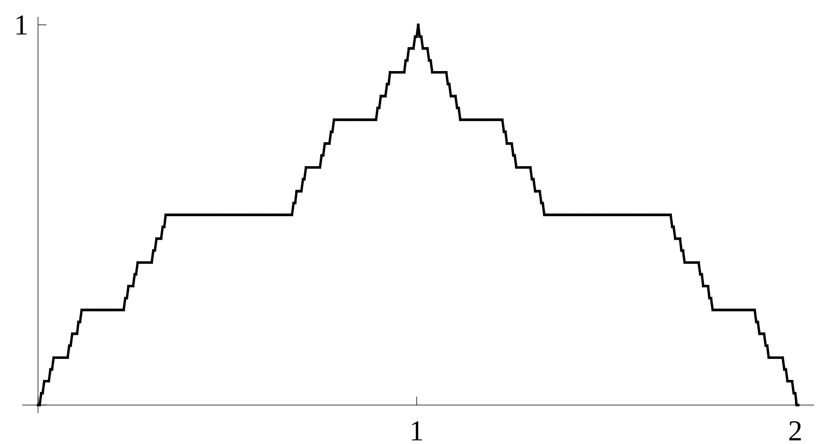

$$\varphi(x) \;=\; \tfrac{1}{2}\varphi(3x) + \tfrac{1}{2}\varphi(3x-1) + \varphi(3x-2) + \tfrac{1}{2}\varphi(3x-3) + \tfrac{1}{2}\varphi(3x-4).$$