On almost periodic causal factorizations

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Outline

Different factorizations

- *LU*-factorization for infinite matrices:

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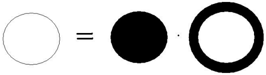
- Wiener-Hopf factorizations:



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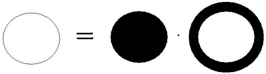

- Factorizations of weighted shifts:

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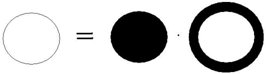

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What do these factorizations have in common?

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For different choices of $\mathcal{T}$ and f with concentrated supp $\hat{f}$, fA serves as the right substitute for a matrix entry or diagonal.

Examples of representations

$$\mathcal{T}(t)A = T(t)AT(-t), \ T : \mathbb{R}^d \to B(X).$$

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More specificaly, "Fourier coefficients" of the function $\mathcal{T}(t)A$ serve as diagonals of A. In the last example this is precisely the case.

Almost periodicity

Definition

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In particular, $\mathcal{T}(t)A \simeq \sum e^{2\pi i\lambda t} A_\lambda$, where the Fourier coefficients A_λ are the eigen-vectors of the representation $\mathcal{T}$.
Weighted shifts that appeared earlier are an example of AP ΨDOs.

Causality

Definition

Let $\mathbb{S} \subset \mathbb{R}^d$ be a closed semigroup such that $-\mathbb{S} \cap \mathbb{S} = \{0\}$. An element $A \in \mathcal{B}$ is *causal with respect to the representation* $\mathcal{T}$ *and semigroup* $\mathbb{S}$, $A \in \mathcal{C}(\mathcal{T}, \mathbb{S})$, if $\Lambda(A, \mathcal{T}) \subseteq \mathbb{S}$. $A \in \mathcal{C}^*$ is anticausal if $\Lambda(A, \mathcal{T}) \subseteq -\mathbb{S}$. $\mathcal{M} = \mathcal{C} \cap \mathcal{C}^*$ is the algebra of memoryless elements, $\mathcal{HC}(\mathcal{HC}^*)$ are ideals of hyper(anti)causal elements.

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Lower triangular matrices and operators that do not shift support of functions to the left are standard examples.

A few more definitions

Definition (Causal factorization)

An invertible element $A \in \mathcal{B}$ *admits a canonical causal factorization* (with respect to $\mathcal{T}$ and $\mathbb{S}$), if $A = A_- A_+$, where $A_+, A_+^{-1} \in \mathcal{C}(\mathcal{T}, \mathbb{S})$ and $A_-, A_-^{-1} \in \mathcal{C}^*(\mathcal{T}, \mathbb{S})$.

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Definition (Decomposing algebras)

An algebra $\mathcal{B}$ is *decomposing* if $\mathcal{B} = \mathcal{C} \oplus \mathcal{H}\mathcal{C}^*$. If $\mathcal{P}$ is the corresponding projection onto $\mathcal{C}$ and $\mathcal{Q} = I - \mathcal{P}$ is the projection onto $\mathcal{H}\mathcal{C}^*$, we let $K(\mathcal{B}) = [\max\{\|\mathcal{P}\|, \|\mathcal{Q}\|\}]^{-1}$.

Factorization in decomposing algebras

Lemma (Follows from [GF71])

Assume that $\mathcal{B}$ is a decomposing Banach algebra, $A \in \mathcal{B}$, and $\|A - I\| < K(\mathcal{B})$. Then A admits a canonical causal factorization $A = A_- A_+$. Moreover, in this case,

$$A_+^{-1} = I - \mathcal{P}M + \mathcal{P}[M\mathcal{P}M] - \mathcal{P}[M\mathcal{P}[M\mathcal{P}M]] + \dots, \qquad (2.1)$$

$$A_-^{-1} = I - \mathcal{Q}M + \mathcal{Q}[[\mathcal{Q}M]M] - \mathcal{Q}[\mathcal{Q}[[\mathcal{Q}M]M]M] + \dots, \qquad (2.2)$$

where $M = A - I$ and the series converge absolutely.

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Proof.

Almost obvious. □

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A Baskakov algebra $\mathcal{B}_\nu = AP_\nu(\mathcal{B}, \mathcal{T})$ is a subalgebra of $AP(\mathcal{B}, \mathcal{T})$ with the norm of $A = \sum A_\lambda$ given by

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A Baskakov algebra is clearly decomposing. It is also inverse closed [BK] if ν is an admissible weight.

Main result

Let $\mathcal{B} = B(H)$ and $\mathcal{B}_\nu = AP_\nu(\mathcal{B}, \mathcal{T})$.

Theorem

Assume that $A \in \mathcal{B}_\nu$ and $\|A - I\|_{B(H)} < 1$. Then A admits a canonical causal factorization $A = A_- A_+$. Moreover, in this case,

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The proof is shorter than [GL72] because we can use a non-commutative Wiener's Lemma.

References

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