On almost periodic causal factorizations

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July 21, 2008

IWOTA 2008, Williamsburg, Virginia
Together with

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Outline

1. Introduction
   - Motivation
   - Operator “matrices” via Beurling spectra
   - Almost periodicity and causality

2. Results

3. Idea of proof

4. References
Different factorizations

- $LU$-factorization for infinite matrices:

\[
\begin{pmatrix}
0 & 1 \\
th & nth
\end{pmatrix}
= 

\begin{pmatrix}
0 & 1 \\
\cdot & \cdot
\end{pmatrix}
\cdot
\begin{pmatrix}
0 & 1 \\
\cdot & \cdot
\end{pmatrix}
\]
Different factorizations

- \textit{LU}-factorization for infinite matrices:

\[
\begin{pmatrix}
\end{pmatrix}
= \begin{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\end{pmatrix}
\]

- Wiener-Hopf factorizations:

\[
\begin{pmatrix}
\end{pmatrix}
= \begin{pmatrix}
\end{pmatrix} \cdot \begin{pmatrix}
\end{pmatrix}
\]

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Different factorizations

- **LU-factorization for infinite matrices:**

\[
\begin{pmatrix}
\vdots \\
\end{pmatrix}
= \begin{pmatrix}
\vdots \\
\end{pmatrix} \begin{pmatrix}
\vdots \\
\end{pmatrix}
\]

- **Wiener-Hopf factorizations:**

\[
\bigcirc = \bullet \cdot \bigcirc
\]

- **Factorizations of weighted shifts:**

\[
\sum_{\lambda \in \mathbb{R}} f_{\lambda}(t)x(t - \lambda) = \sum_{\lambda \geq 0} \sum_{\mu \leq 0} g_{\lambda}(t) h_{\mu}(t - \lambda)x(t - \lambda - \mu)
\]
Different factorizations

- *LU*-factorization for infinite matrices:

  \[
  \begin{pmatrix}
  \mathbf{L} & \mathbf{U} \\
  \end{pmatrix}
  =
  \begin{pmatrix}
  \mathbf{L} & \mathbf{0} \\
  \end{pmatrix}
  \begin{pmatrix}
  \mathbf{U} & \mathbf{0} \\
  \end{pmatrix}
  \]

- Wiener-Hopf factorizations:

  \[
  \begin{pmatrix}
  \mathbf{A} \\
  \end{pmatrix}
  =
  \begin{pmatrix}
  \mathbf{A}^{+} & \mathbf{0} \\
  \end{pmatrix}
  \begin{pmatrix}
  \mathbf{A}^{-} & \mathbf{0} \\
  \end{pmatrix}
  \]

- Factorizations of weighted shifts:

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  What do these factorizations have in common?
Notation

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- $fA = \mathcal{T}(f)A = \int f(t)\mathcal{T}(-t)Adt$ -- $L^1(\mathbb{R}^d)$-module structure;

Definition (Beurling Spectrum)

$\Lambda(A, \mathcal{T}) = \{ \lambda \in \mathbb{R}^d : fA = 0 \Rightarrow \hat{f}(\lambda) = 0 \}$

For different choices of $\mathcal{T}$ and $f$ with concentrated $\hat{f}$, $fA$ serves as the right substitute for a matrix entry or diagonal.
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For different choices of $\mathcal{T}$ and $f$ with concentrated supp $\hat{f}$, $fA$ serves as the right substitute for a matrix entry or diagonal.
Examples of representations

\[ T(t)A = T(t)AT(-t), \quad T : \mathbb{R}^d \to B(X). \]

- Modulation \( T = M : M(t)x(s) = e^{2\pi ist}x(s), \)
  \( \Lambda(x, M) = \text{supp } x; \)
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  \[ S(t)x(s) = x(s - t), \quad \Lambda(x, M) = \text{supp } \hat{x}; \]
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- **Based on a resolution of the identity** \( T = U \):
  \[ U(t)x = \sum e^{2\pi int}E(n)x, \quad \Lambda(x, U) = \{ n \in \mathbb{Z}^d : E(n)x \neq 0 \}. \]
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- Based on a resolution of the identity \( T = U: \)
  \[ U(t)x = \sum e^{2\pi i nt}E(n)x, \; \Lambda(x, U) = \{ n \in \mathbb{Z}^d : E(n)x \neq 0 \}. \]

More specifically, “Fourier coefficients” of the function \( T(t)A \) serve as diagonals of \( A. \) In the last example this is precisely the case.
Definition

An element $A \in \mathcal{B}$ is almost periodic with respect to the representation $\mathcal{T}$, $A \in \text{AP}(\mathcal{B}, \mathcal{T})$, if the function $\mathcal{T}(t)A$ is (Bohr) almost periodic.
Almost periodicity

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In particular, $\mathcal{T}(t)A \sim \sum e^{2\pi i \lambda t} A_\lambda$, where the Fourier coefficients $A_\lambda$ are the eigen-vectors of the representation $\mathcal{T}$. Weighted shifts that appeared earlier are an example of AP $\Psi$DOs.
Causality

Definition

Let $S \subset \mathbb{R}^d$ be a closed semigroup such that $-S \cap S = \{0\}$. An element $A \in B$ is causal with respect to the representation $T$ and semigroup $S$, $A \in C(T, S)$, if $\Lambda(A, T) \subseteq S$. $A \in C^*$ is anticausal if $\Lambda(A, T) \subseteq -S$. \( M = C \cap C^* \) is the algebra of memoryless elements, $HC(HC^*)$ are ideals of hyper(anti)causal elements.
Definition

Let $\mathbb{S} \subset \mathbb{R}^d$ be a closed semigroup such that $-\mathbb{S} \cap \mathbb{S} = \{0\}$. An element $A \in \mathcal{B}$ is causal with respect to the representation $\mathcal{T}$ and semigroup $\mathbb{S}$, $A \in C(\mathcal{T}, \mathbb{S})$, if $\Lambda(A, \mathcal{T}) \subseteq \mathbb{S}$. $A \in C^*$ is anticausal if $\Lambda(A, \mathcal{T}) \subseteq -\mathbb{S}$. $\mathcal{M} = C \cap C^*$ is the algebra of memoryless elements, $\mathcal{HC}(\mathcal{HC}^*)$ are ideals of hyper(anti)causal elements.

Lower triangular matrices and operators that do not shift support of functions to the left are standard examples.
A few more definitions

**Definition (Causal factorization)**

An invertible element $A \in B$ admits a canonical causal factorization (with respect to $T$ and $S$), if $A = A_- A_+$, where $A_+, A_+^{-1} \in C(T, S)$ and $A_-, A_-^{-1} \in C^*(T, S)$. 
Definition (Causal factorization)

An invertible element $A \in \mathcal{B}$ admits a canonical causal factorization (with respect to $\mathcal{T}$ and $\mathcal{S}$), if $A = A_- A_+$, where $A_+, A_+^{-1} \in C(\mathcal{T}, \mathcal{S})$ and $A_-, A_-^{-1} \in C^*(\mathcal{T}, \mathcal{S})$.

Definition (Decomposing algebras)

An algebra $\mathcal{B}$ is decomposing if $\mathcal{B} = \mathcal{C} \oplus \mathcal{HC}^*$. If $\mathcal{P}$ is the corresponding projection onto $\mathcal{C}$ and $\mathcal{Q} = I - \mathcal{P}$ is the projection onto $\mathcal{HC}^*$, we let $K(\mathcal{B}) = \left[\max\{\|\mathcal{P}\|, \|\mathcal{Q}\|\}\right]^{-1}$. 

Ilya Krishtal
On almost periodic causal factorizations
Lemma (Follows from [GF71])

Assume that $\mathcal{B}$ is a decomposing Banach algebra, $A \in \mathcal{B}$, and $\|A - I\| < K(\mathcal{B})$. Then $A$ admits a canonical causal factorization $A = A_- A_+$. Moreover, in this case,

$$A_+^{-1} = I - PM + P[MPM] - P[MP[MPM]] + \ldots, \quad (2.1)$$


where $M = A - I$ and the series converge absolutely.
Factorization in decomposing algebras

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$$A_-^{-1} = I - QM + Q[[QM]M] - Q[Q[[QM]M]M] + \ldots,$$  \hspace{1cm} (2.1)

$$A_+^{-1} = I - PM + P[MPM] - P[MP[MPM]] + \ldots,$$  \hspace{1cm} (2.2)

where $M = A - I$ and the series converge absolutely.

Proof.

Almost obvious.
Baskakov Algebras

We want to weaken the condition $\|A - I\| < K(B)$ as in [GL72] in a special class of algebras. In this way we extend the “periodic” results of [GL72] to an AP setting.
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**Definition**

A Baskakov algebra $B_{\nu} = AP_{\nu}(B, T)$ is a subalgebra of $AP(B, T)$ with the norm of $A = \sum A_\lambda$ given by

$$\|A\|_{\nu} = \sum \|A_\lambda\| \nu(\lambda),$$

where $\nu$ is a “nice” weight.
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where \( \nu \) is a “nice” weight.

A Baskakov algebra is clearly decomposing. It is also inverse closed [BK] if \( \nu \) is an admissible weight.
Let $\mathcal{B} = \mathcal{B}(H)$ and $\mathcal{B}_\nu = AP_\nu(\mathcal{B}, T)$.

**Theorem**

Assume that $A \in \mathcal{B}_\nu$ and $\|A - I\|_{\mathcal{B}(H)} < 1$. Then $A$ admits a canonical causal factorization $A = A_- A_+$. Moreover, in this case,

$$A_+^{-1} = I - P M + P[MPM] - P[MP[MPM]] + \ldots , \quad (2.3)$$


where $M = A - I$ and the series converge in $\mathcal{B}_\nu$ in norm.
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Examples and extensions

Positive definite invertible $A \in \mathcal{B}_\nu$ admit a spectral causal factorization;
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- Positive definite invertible localized time-frequency shifts admit spectral causal factorizations w.r.t. half-planes;
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- Positive definite invertible $\Psi$DOs in Sjöstrand class admit spectral $\epsilon$-causal factorizations w.r.t. half-planes.
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It is enough to solve equations

\[ AX_+ = I + X_ - \quad \text{and} \quad (I + Y_-)A = Y_+. \]
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- Lift the equations to an $H$-valued Besicovitch space, where projections that correspond to $P$ and $Q$ become orthogonal.
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- Solve the lifted equations similar to [GL72].
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- Get back to the original equations.
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The proof is shorter than [GL72] because we can use a non-commutative Wiener’s Lemma.
References

