Reproducing systems and their applications

by

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Lecture 1:

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OUTLINE

riteria “real world” problems:

- Image processing;
- Sleep staging;
- Netflix problem, etc.

Reproducing systems and mathematical problems behind:

- Fourier and Zak Transforms, sampling formulae;
- Bases, frames, resolutions of the identity, etc;
- Gabor (Weyl-Heisenberg) frames;
- Wavelets.
Motivational problems

Image processing

Compression

Original image

1 level Haar wavelet decomposition

2 level Haar wavelet decomposition

3 level Haar wavelet decomposition
Motivational problems

Image processing

- Denoising

Target (unknown)  Input image

Isotropic  Anisotropic
Motivational problems

Image processing

- Inpainting

GREYCstoration
Motivational problems

Sleep staging

Sleep stages
- Stage N1 (4-7 Hz)
- Stage N2 (11-16 Hz)
- Stage N3 (0.5-2 Hz)
- REM sleep
Motivational problems

Netflix Problem

New Releases on DVD

Which movies to suggest to customers based on their previous selections?
Reproducing systems

**Standard notation**

- $L^p(\mathbb{R}^d)$, $L^p(\mathbb{T}^d)$ – Hilbert spaces of (equivalence classes of) $p$-integrable functions on $\mathbb{R}^d$ or $\mathbb{T}^d$, respectively;

- **Bracket product:**

  $$\left[\varphi, \psi\right](x) = \sum_{k \in \mathbb{Z}} \varphi(x - k)\psi(x - k)$$

- $\langle \psi \rangle$ – Principal Shift-invariant space generated by $\psi$.

- **Norm equality:**

  $$\|f\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx = \sum_{k \in \mathbb{Z}} \int_0^1 |f(x - k)|^2 \, dx = \int_0^1 \left[ f, f \right](x) \, dx = \left\| f, f \right\|_{L^1(\mathbb{T})}$$
Reproducing systems

Zak and dual Zak transforms

- Zak and dual Zak transforms
  \[(\mathcal{Z} f)(x, \xi) \equiv \varphi(x, \xi) := \sum \limits_{k \in \mathbb{Z}} f(x + k)e^{-2\pi ik\xi},\]
  \[(\mathcal{\tilde{Z}} f)(x, \xi) \equiv \tilde{\varphi}(x, \xi) := \sum \limits_{\ell \in \mathbb{Z}} f(\xi + \ell)e^{2\pi i\ell x} = \varphi(\xi, -x)\]

- Map \(L^2(\mathbb{R})\) isometrically onto \(L^2(\mathbb{T}^2)\) since
  \[\left[ f, f \right](x) = \int \limits_{0}^{1} \left| \varphi(x, \xi) \right|^2 d\xi \quad a.e.\]

- Inverse Z-transforms:
  \[\int \limits_{0}^{1} (\mathcal{Z}(f))(x, \xi) d\xi = f(x), \quad \int \limits_{0}^{1} (\mathcal{\tilde{Z}}(f))(x, \xi) dx = f(\xi).\]
Reproducing systems

**Zak and Fourier transforms**

**Fourier transform:**

\[
(\mathcal{F}(f))(\xi) \equiv \hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi ix}\xi\,dx
\]

**A unitary map to relate Zak and Fourier transforms:**

\[
(\mathcal{U}\varphi) := e^{-2\pi ix}\xi\varphi(x, \xi), \quad \left(\widetilde{\mathcal{Z}}^{-1}\mathcal{U}\mathcal{Z}\right) = \mathcal{F}, \quad \left(\mathcal{Z}^{-1}\mathcal{U}^*\mathcal{Z}\right) = \mathcal{F}^{-1}.
\]

\[
\left(\mathcal{Z}^{-1}\mathcal{U}\mathcal{Z}f\right)(\xi) = \int_{\mathbb{R}} f(x+k)e^{-2\pi i(x+k)\xi}\,dx = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix}\xi\,dx = (\mathcal{F}f)(\xi)
\]

**Poisson Summation formula:**

\[
\sum_{\ell=-\infty}^{\infty} f(x + \ell)e^{-2\pi i\xi(x + \ell)} = (\mathcal{U}\mathcal{Z})f = \left(\mathcal{Z}\mathcal{F}\right)f = \sum_{\ell=-\infty}^{\infty} \hat{f}(\xi + \ell)e^{-2\pi i\ell x}.
\]

G. L. Weiss
Suppose $f$ is band limited and $\text{supp} \ (\mathcal{F}f) = [-1/2, 1/2]$.

Poisson summation formula for $x = 0$, $\xi$ in $[-1/2, 1/2]$:

$$\sum_{\ell=-\infty}^{\infty} f(\ell)e^{-2\pi i \ell \xi} = \hat{f}(\xi)$$

Kotel’nikov-Shannon-Whittaker formula:

$$f(x) = \int_{-1/2}^{1/2} \hat{f}(\xi)e^{2\pi i \xi x} d\xi = \int_{-1/2}^{1/2} \sum_{\ell=-\infty}^{\infty} f(\ell)e^{2\pi i \xi (x-\ell)} d\xi = \sum_{\ell=-\infty}^{\infty} f(\ell)\text{sinc}(x-\ell)$$

$$\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

G. L. Weiss
Consider \( \psi \) in \( L^2(\mathbb{R}) \). Pick a representative in the class such that \( [\psi, \psi](x) < \infty \) everywhere, and

\[
\psi(k) = \begin{cases} 
1, & k = 0 \\
0, & k \neq 0,
\end{cases}
\]

\( k \) – integer.

Let \( c = \{c_k\} \) in \( l^2(\mathbb{Z}) \) and

\[
f_c(x) = \sum_{\ell=-\infty}^{\infty} c_\ell \psi(x - \ell)
\]

Convergence above is absolute and unconditional.

Since \( f_c(k) = c_k \), we get the general sampling formula:

\[
f_c(x) = \sum_{\ell=-\infty}^{\infty} f_c(\ell) \psi(x - \ell)
\]

G. L. Weiss
Reproducing systems

**Bases, frames, etc.**

- Completeness, linear independence, conditionality.
- Bases: orthonormal, Riesz, Shauder...
- Frames: tight, equal norm, Parseval...
- Resolutions of the identity
- Fusion frames (frames of subspaces)
Reproducing systems

**Important unitary reps**

- **Translations** $T_k: \mathbb{L}^2(\mathbb{R}) \to \mathbb{L}^2(\mathbb{R})$.

  $$T_k f(x) = f(x - k), \quad T_k = T^k = T(k)$$

- **Modulations** $M_b: \mathbb{L}^2(\mathbb{R}) \to \mathbb{L}^2(\mathbb{R})$.

  $$M_b f(x) = e^{2\pi ibx} f(x), \quad M_\ell = M^\ell = M(\ell)$$

- **Dilations** $D_a: \mathbb{L}^2(\mathbb{R}) \to \mathbb{L}^2(\mathbb{R})$.

  $$(D_a f)(x) = 1/\sqrt{|a|} f(x/a)$$
Gabor systems

- **Gabor systems generated by** $g$ in $L^2(\mathbb{R})$:
  \[ G_{a,b}(g) = \{ M_{b\ell} T_{ak} g, \ell, k \in \mathbb{Z} \} \]

- **When is the system $G$ a Riesz basis, a frame?**

- **What windows $g$ are more suitable than others?**

- **Time-frequency localization, Uncertainty Principle.**

- **Weyl-Heisenberg systems.**
Reproducing systems

Affine systems

- **Affine systems generated by** $\psi$ **in** $L^2(\mathbb{R})$: 

  $$\Psi_a(\psi) = \left\{ D^j_a T_k \psi, j, k \in \mathbb{Z} \right\}$$

- **When is the system** $\Psi$ **a Riesz basis, a frame?**

- **What generators** $\psi$ **are more suitable than others?**

- **Higher dimensions.**

- **Preview of lecture 2.**