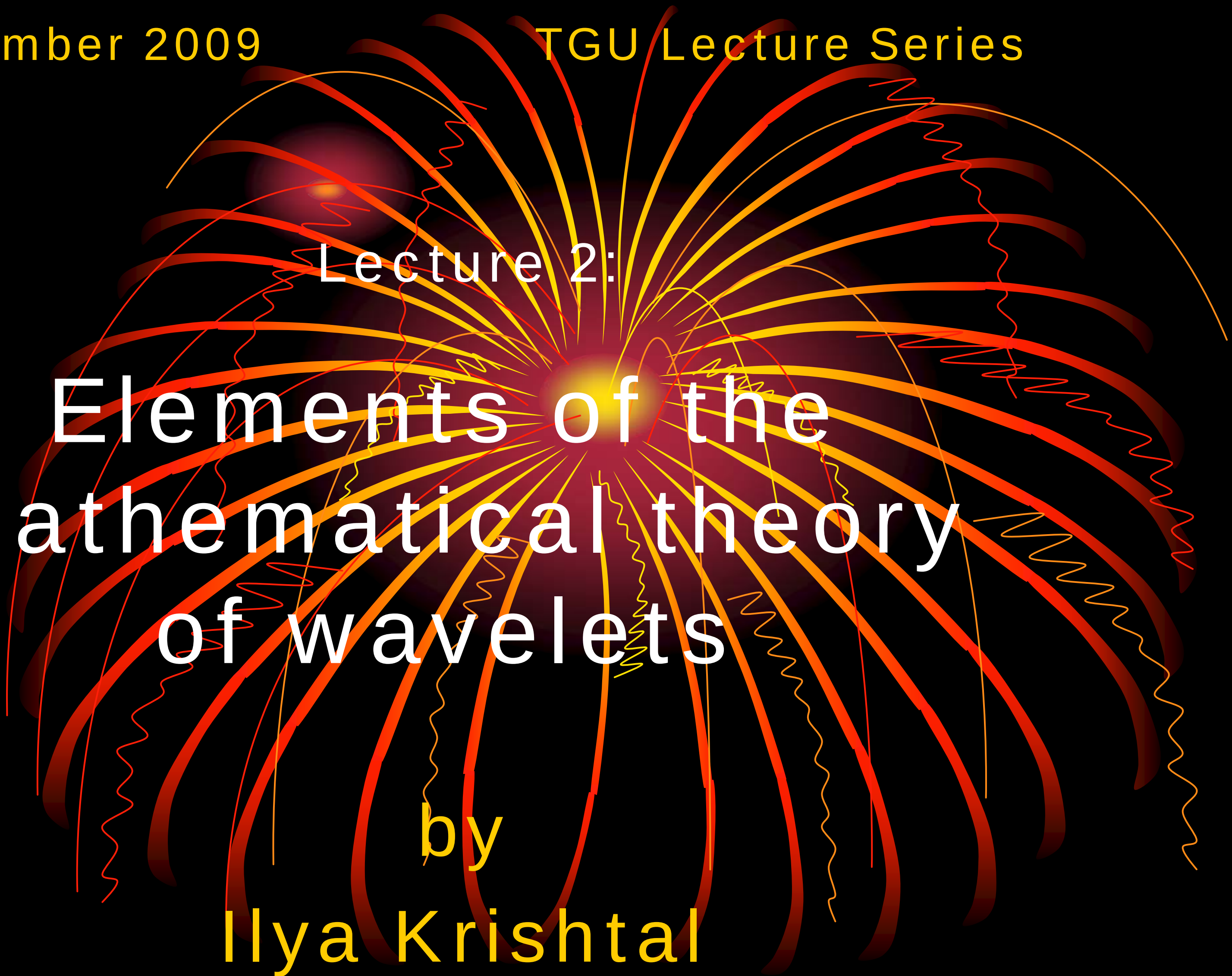
December 2009 TGU Lecture Series



Lecture 2:

Elements of the mathematical theory of wavelets

by

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OUTLINE

- Classical dyadic wavelets.
    - Definition and examples;
    - Types of wavelets (MRA, MSF);
    - Elements of the MRA method.
- Higher dimensions.
    - Tensor product constructions;
    - Unitary extension principle;
    - Composite wavelets;
    - “Blubbery blob”.

Important unitary reps

- ⌘ **Translations T_k: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.**

$$T_k f(x) = f(x-k), \quad T_k = T^k = T(k)$$

- ⌘ **Modulations M_b: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.**

$$M_b f(x) = e^{2\pi i b x} f(x), \quad M_\ell = M^\ell = M(\ell)$$

- ⌘ **Dilations D_a: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.**

$$(D_a f)(x) = 1/\sqrt{|a|} f(x/a)$$

Affine systems, wavelets

- **Dyadic affine systems generated by ψ in $L^2(\mathbf{R})$:**

$$\Psi_2(\psi) = \left\{ D_2^j T_k \psi = \psi(2^{-j} \cdot - k),\ j,k \in \mathbf{Z} \right\}$$

- **ψ in $L^2(\mathbf{R})$ is an orthonormal wavelet if the above Affine system is an othonormal basis for $L^2(\mathbf{R})$.**

- **Shannon wavelet:** $\hat{\psi}(\xi) = e^{\pi i \xi} \chi_I(\xi),\ I = [-1, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1].$

- **Haar wavelet:** $\psi(x) = \chi_{(\frac{1}{2},1]}(x) - \chi_{(0,\frac{1}{2}]}(x),$

- **Exercise:** $\hat{\psi}(\xi) = ?$ **Express it using** $\operatorname{sinc}(\xi) = \dfrac{\sin \xi}{\xi}.$

Characterization equations

ψ **in** $L^2(\mathbf{R})$ **is an orthonormal wavelet if and only if the following conditions hold:**

- **Calderon condition:**

$$\sum_{j\in\mathbf{Z}}\left|\hat{\psi}(2^j\xi)\right|^2=1,\ \text{ for a.e. } \xi\in\mathbf{R}.$$

- **"T-q" condition:**

$$\sum_{j\in\mathbf{Z}_+}\hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+q))}=0,\ \text{ for all odd } q \text{ and a.e. } \xi\in\mathbf{R}.$$

- **Exercise: Verify that the above conditions hold for Shannon and Haar wavelet.**

Types of dyadic wavelets

- **MSF wavelets (wavelet sets).**
- **MRA wavelets.**
- **Daubechies wavelets.**
- **Lemarié-Meyer wavelets.**
- **Journé wavelet.**

Multiresolution Analysis (MRA)

MRA consists of a (nested) sequence of subspaces V_j in $L^2(\mathbb{R})$ satisfying:

- $V_j \subset V_{j+1}$ **for all** j **in** $\mathbb{Z}$;
- $f \in V_j$ **if and only if** $D_2 f \in V_{j-1}$ **for all** $j \in \mathbb{Z}$;
- $\bigcap_{j\in\mathbf{Z}} V_j = \{0\};$
- $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R});$
- **There exists a function** $\varphi \in V_0$ **such that** $\{T_k\, \varphi,\ k \in \mathbb{Z}\}$ **is an othonormal basis for** V_0**.**

Scaling equation, filters

- **Scaling equation (time domain):**

$$\varphi(\tfrac{x}{2}) = \sum_{k\in\mathbf{Z}} c_k \varphi(x-k).$$

- **Scaling equation (frequency domain):**

$$\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$$

- **Smith-Barnwell equation:**

$$|m_0|^2(\xi) + |m_0|^2(\xi + \tfrac{1}{2}) = 1, \text{ for a.e. } \xi.$$

- **Exercise: Prove the S.-B. equation using** $[\hat{\varphi},\hat{\varphi}](\xi) = 1, \text{ for a.e. } \xi.$

MRA wavelet construction

- **High-pass filter:**

$$m_1(\xi) = e^{\pi i \xi}\overline{m_0(\xi + \tfrac{1}{2})}.$$

- **Wavelet from scaling function:**

$$\hat{\psi}(2\xi) = m_1(\xi)\hat{\varphi}(\xi)$$

- **Fact:**

$$\psi \in W_0, V_1 = V_0 \oplus W_0.$$

- **Exercise: Prove the above fact by showing** $[\hat{\psi}, \hat{\varphi}](\xi) = 0$, for a.e. ξ.

Tensor product construction

- **Let ϕ and ψ be 1-d (dyadic) scaling function and wavelet.**
- **2-d "tensor product" scaling function and wavelets:**

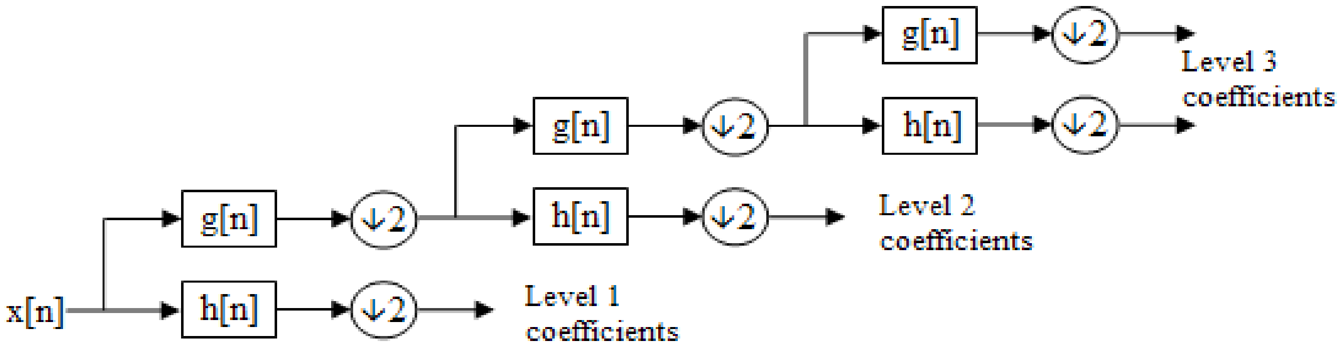
$$\Phi(\eta,\xi)=\varphi(\eta)\varphi(\xi), \quad \Psi_1(\eta,\xi)=\psi(\eta)\varphi(\xi),$$

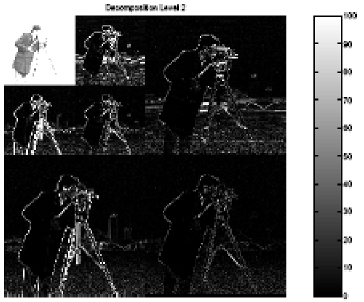
$$\Psi_2(\eta,\xi)=\varphi(\eta)\psi(\xi), \quad \Psi_3(\eta,\xi)=\psi(\eta)\psi(\xi).$$

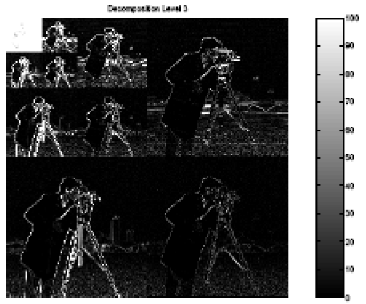
- **Exercise: Prove that the affine system generated by the three 2-d wavelet is indeed an o.n.b.**

Image processing

- Wavelet decomposition:



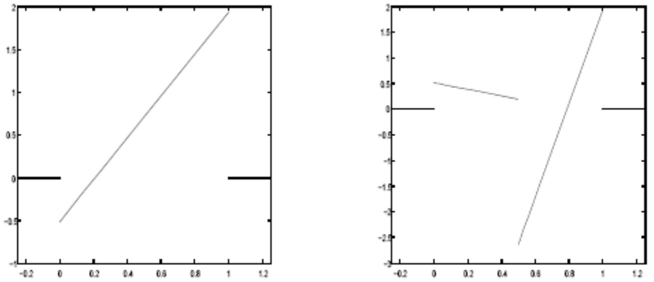




2 level Haar wavelet decomposition

3 level Haar wavelet decomposition

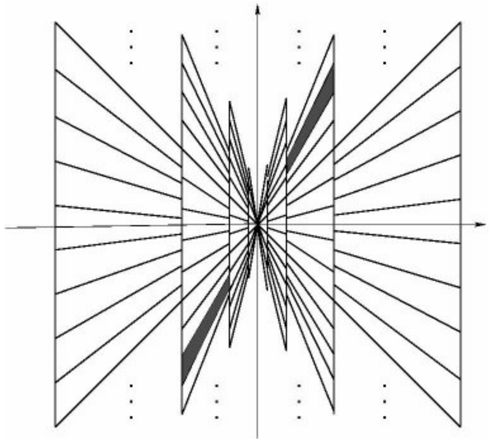
Composite wavelets



- **Composite affine systems generated by ψ in $L^2(\mathbf{R}^d)$:**

$$\Psi_{a,b}(\psi) = \left\{ D_a^j D_b T_k \psi,\ j,k \in \mathbf{Z} \right\}$$

- **Shearlets (http://www.shearlet.org/):**



$$B = \left\{ b = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} ; k \in \mathbb{Z} \right\}$$

- **Piece-wise linear 1-d example:**

$$\varphi(x) = \begin{cases} \frac{1}{\sqrt{2}} \left[2\sqrt{3}x + (1-\sqrt{3}) \right] & \text{for} \quad x \in [0,1], \\ 0 & \text{for} \quad x \notin [0,1] \end{cases}$$

“Blubbery Blob”

- Low pass filter matrix in 1-d:

$$M_0(\xi) = \frac{1}{2}\begin{pmatrix} \alpha + \beta e^{-2\pi i\xi} & \gamma e^{-2\pi i\xi} + \delta e^{-4\pi i\xi} \\ \gamma e^{2\pi i\xi} + \delta e^{4\pi i\xi} & \alpha + \beta e^{2\pi i\xi} \end{pmatrix}$$

- Exercise: Compute the “tensor-product” low-pass filter matrix for dilation [0,2; 1,0].