Slanted matrices, Sampling, and Banach frames

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Outline

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What if we change p?

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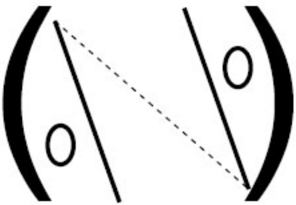

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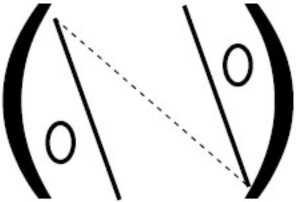

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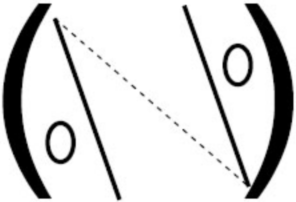

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Does (1.2) remain valid **for all** p?

Main Result

Theorem (ABK)

Let $\mathbb{A}$ *be a matrix with sufficient off-slant decay and satisfying* (1.2) *for some* $p \in [1, \infty]$. *Then* $\mathbb{A}$ *satisfies* (1.2) *for all* $p \in [1, \infty]$. *Moreover, a universal lower bound exists and can be estimated.*

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Proof.

$2 \to p$. Very easy in a Hilbert space:

$$\|c\|^2 \sim \langle \mathbb{A}c, \mathbb{A}c \rangle = \langle \mathbb{A}^*\mathbb{A}c, c \rangle$$

implies invertibility of $\mathbb{A}^*\mathbb{A}$ in ℓ^2, invertibility in ℓ^p follows from Wiener's Lemma, and, hence, $\mathbb{A}$ is left invertible in all ℓ^p.

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General case. Very hard: over 5 pages of proof. Involves $p \to \infty$, $\infty \to p$, and Cesaro means. □

Conclusions

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For second order: $\frac{1}{2}(1-\gamma^2(X))$ in ℓ^∞.

References

AAK E. Acosta-Reyes, A. Aldroubi, and I. Krishtal, On Stability of Sampling-Reconstruction Models, submitted (2007).

ABK A. Aldroubi, A. Baskakov, and I. Krishtal, SLANTED MATRICES, BANACH FRAMES, AND SAMPLING, submitted (2007).

AK A. Aldroubi and I. Krishtal, Robustness of sampling and reconstruction and Beurling-Landau-type theorems for shift invariant spaces, *Appl. Comput. Harmon. Anal.*, **20** (2006), no.2, 250–260.

The papers are available via http://www.math.niu.edu/~krishtal/ or from ArXiV.