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Some Simple Haar-Type Wavelets in Higher Dimensions

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ABSTRACT. *An orthonormal wavelet system in $\mathbb{R}^d$, $d \in \mathbb{N}$, is a countable collection of functions $\{\psi^\ell_{j,k}\}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, $\ell = 1, \dots, L$, of the form*

$$\psi^\ell_{j,k}(x) = |\det a|^{-j/2}\psi^\ell\big(a^{-j}x - k\big) \equiv \big(D_{a^j}\, T_k\, \psi^\ell\big)(x)$$

that is an orthonormal basis for $L^2(\mathbb{R}^d)$, where $a \in \mathrm{GL}_d(\mathbb{R})$ is an expanding matrix. The first such system to be discovered (almost 100 years ago) is the Haar system for which $L = d = 1$, $\psi^1(x) = \psi(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)$, $a = 2$. It is a natural problem to extend these systems to higher dimensions. A simple solution is found by taking appropriate products $\Phi(x_1, x_2, \dots, x_d) = \varphi_1(x_1)\varphi_2(x_2)\dots\varphi_d(x_d)$ of functions of one variable. The obtained wavelet system is not always convenient for applications. It is desirable to find "nonseparable" examples. One encounters certain difficulties, however, when one tries to construct such MRA wavelet systems. For example, if $a = \left(\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right)$ is the quincunx dilation matrix, it is well-known (see, e.g., [5]) that one can construct nonseparable Haar–type scaling functions which are characteristic functions of rather complicated fractal–like compact sets. In this work we shall construct considerably simpler Haar–type wavelets if we use the ideas arising from "composite dilation" wavelets. These were developed in [7] and involve dilations by matrices that are products of the form $a^j b$, $j \in \mathbb{Z}$, where $a \in \mathrm{GL}_d(\mathbb{R})$ has some "expanding" property and b belongs to a group of matrices in $\mathrm{GL}_d(\mathbb{R})$ having $|\det b| = 1$.

1. Introduction with a description of composite dilation wavelets

We begin by describing a special class of wavelets we call *composite dilation wavelets* (see [7]). Let us introduce the following notation: If $c \in \mathrm{GL}_d(\mathbb{R})$, the *dilation* D_c is the operator that maps a function f on $\mathbb{R}^d$ to the function $(D_c f)(x) = |\det c|^{-1/2} f(c^{-1}x)$. If Γ is the lattice in $\mathbb{R}^d$ obtained by applying a matrix $M \in \mathrm{GL}_d(\mathbb{R})$ to the points of the integer lattice $\mathbb{Z}^d$ ($\Gamma = M\mathbb{Z}^d$), the *translation* by $\gamma \in \Gamma$ is the operator T_γ that maps a function f on $\mathbb{R}^d$ to the function $(T_\gamma f)(x) = f(x - \gamma)$. Each of these operators is unitary when it is restricted to $L^2(\mathbb{R}^d)$. If $C \subset \mathrm{GL}_d(\mathbb{R})$ and $\Psi = \{\psi^1, \psi^2, \dots, \psi^L\} \subset L^2(\mathbb{R}^d)$, then the *affine system* $\{D_c T_\gamma \psi^\ell : c \in C, \gamma \in \Gamma, \ell = 1, 2, \dots, L\}$ is called an *orthonormal wavelet system* or, simply, a *(multi)wavelet* associated with the dilation set C and the lattice Γ, if and only if it is

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an orthonormal basis for $L^2(\mathbb{R}^d)$. We are particularly interested in cases where C is the product of two sets $A, B \subset \mathrm{GL}_d(\mathbb{R})$, where B is a group of matrices with $|\det b| = 1$ for all $b \in B$ and (some of) the elements of A have an "expanding" property. As shown in [7] none of these matrices need to be expanding in the sense that each proper value has absolute value strictly bigger than 1; in fact, each $a \in A$ can have $|\det a| \leq 1$ and, yet, there are associated affine systems which are orthonormal bases for $L^2(\mathbb{R}^d)$. In this article, the set B will be a finite group. We will also assume that matrices preserve the lattice Γ, that is, $B(\Gamma) = \Gamma$.

In what follows, we will construct an MRA structure that is a bit more general than the "classical" one. Let us recall that the classical MRA associated with a sequence of dilations $\{a^j\}_{j\in\mathbb{Z}} = A$, where a is an expanding matrix, is an increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ such that

(M1) $V_j = D_{a^{-j}} V_0$;

(M2) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$;

(M3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(M4) there exists $\varphi \in V_0$ such that $\{T_\gamma \varphi\}$, $\gamma \in \Gamma$, is an orthonormal basis for V_0.

The last property states, in particular, that V_0 is the shift invariant space generated by the function φ.

In our "composite dilation" case the notion of shift-invariance is replaced by a more general concept. Since $B(\Gamma) = \Gamma$, the operators generated by the dilations D_b, $b \in B$, and the translations T_γ, $\gamma \in \Gamma$, form a group. In fact, it is easily verified that

$$(D_c T_\tau)(D_b T_\gamma) f = D_{cb} T_{b^{-1}\tau+\gamma} f \ . \tag{1.1}$$

This allows us to define the operation on the Cartesian product $B \times \Gamma$:

$$(c, \tau) \cdot (b, \gamma) = \left(cb, b^{-1}\tau + \gamma\right), \tag{1.2}$$

and we obtain a new group that we denote by BΓ(it is, in fact, the semi-direct product of B and Γ). This leads us to the notion of BΓ-*invariant spaces* which are the closed subspaces V of $L^2(\mathbb{R}^d)$ such that $D_b T_\gamma f \in V$ whenever $f \in V$, $b \in B$, and $\gamma \in \Gamma$. This notion, in turn, leads us to the following more general version of (M4):

(M4′) There exists $\varphi \in V_0$ such that $\{D_b T_\gamma \varphi\}$, $b \in B$ and $\gamma \in \Gamma$, is an orthonormal basis for V_0.

Thus, in particular, V_0 is the BΓ-invariant space generated by φ.

The more general notion of a *multi-resolution analysis associated with the dilation sets* $A = \{a^j\}_{j\in\mathbb{Z}}$ *and the group* $B \subset \mathrm{GL}_d(\mathbb{R})$, or AB-MRA, is an increasing collection $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ that satisfies the conditions (M1), (M2), (M3), and (M4′).

Such a composite MRA can also be considered as a traditional affine MRA associated with the dilations in A (A-MRA); however, the space V_0 is not generated by the Γ-translations of a *single* scaling function φ. In fact, the generators are, then, the set of functions $D_b\varphi$, $b \in B$. This is clearly the case by observing that $D_b T_\gamma \varphi = T_{b\gamma} D_b \varphi$ and $B(\Gamma) = \Gamma$.

The purpose of this article is to present the construction of several examples of compactly supported multiwavelets $\Psi = \{\psi^1, \psi^2, \ldots, \psi^\ell\}$ such that the affine system

$$\left\{D_{a^j} D_b T_\gamma \psi^\ell : \ j \in \mathbb{Z},\ b \in B,\ \gamma \in \Gamma,\ \ell = 1, 2, \ldots, L\right\} \tag{1.3}$$

is an orthonormal basis for $L^2(\mathbb{R}^d)$. We will do so for several finite groups B and our goal includes the requirement that each ψ^ℓ is a finite linear combination of characteristic functions of very simple sets (for example, triangles). A consequence of this will be the fact that the ψ^ℓ's and φ are related by rather simple equations involving polynomial filters and, consequently, are good candidates for applications.

Several authors [1, 4, 5, etc.] constructed nonseparable Haar-like wavelets in several dimensions. In the abstract we mentioned a Haar-like wavelet associated with the quincunx dilations whose scaling function is the characteristic function of a rather complicated fractal set known as the "twin dragon." We shall show that the introduction of a rather simple group B allows us to construct a considerably simpler Haar wavelet of composite dilation type in two dimensions when $A = \{a^j : j \in \mathbb{Z}\}$ is generated by the quincunx matrix. We will construct other examples having these features in two dimensions. It is our hope that these examples will lead to the discovery of smoother compactly supported wavelets that are not separable in higher dimensions and enjoy comparable simplicity. Some other related works are [1, 2, 3, 4, 8, 10, 11]. Finally, we would like to thank N. Arcozzi, J. Blanchard, D. Labate, D. Speegle, and W. Zhu for making very helpful observations when we presented this work to them.

2. A very simple Haar-like wavelet associated with the quincunx dilation

Let $a = q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ be the quincunx matrix and $B = \{b_i : i = 1, 2, \ldots 7\}$ the group of symmetries of the unit square, that is,

$$b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ b_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ b_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ b_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $b_i = -b_{i-4}$ for $i = 4, 5, 6, 7$. Let R_0 be the triangular region with vertices $(0, 0)$, $(\frac{1}{2}, 0)$, and $(\frac{1}{2}, \frac{1}{2})$ and $R_i = b_i R_0$, for $i = 1, 2, \ldots, 7$ (see Figure 1). If $\varphi = 2\sqrt{2}\chi_{R_0}$, we claim that the system $\{D_b T_k \varphi : b \in B,\ k \in \mathbb{Z}^2\}$ is an orthonorнal basis for its closed linear span which we denote by V_0.

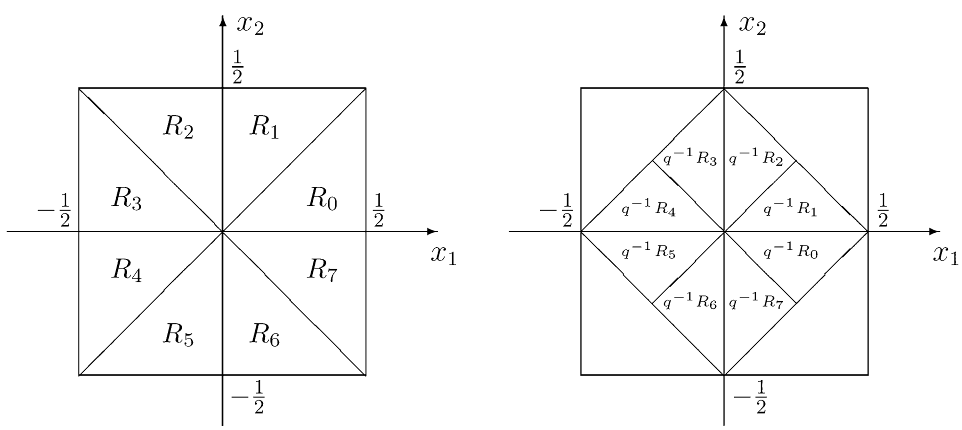

FIGURE 1 The fundumental region R_0 and its images under B and q^{-1}.

It is clear that V_0 is the subspace of $L^2(\mathbb{R}^2)$ consisting of all square integrable functions that are constant on each $\mathbb{Z}^2$-translate of the triangles R_i, $i = 1, 2, \ldots, 7$. Let us now consider the spaces $V_j = D_{q^{-j}} V_0$, $j \in \mathbb{Z}$. The space V_1 consists of all functions in $L^2(\mathbb{R}^2)$ that are constant on each $q^{-1}\mathbb{Z}^2$-translate of the triangles $q^{-1}R_i$, $i = 0, 1, \ldots, 7$ (see Figure 1). Thus, clearly, $V_0 \subset V_1$, and, consequently, $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$. Now, it is straightforward to see that the spaces V_j, $j \in \mathbb{Z}$, form an AB-MRA with φ as a scaling function. As we mentioned above, another point of view is to consider the column vector

$$\Phi = \begin{bmatrix} D_{b_0}\varphi \\ D_{b_1}\varphi \\ \vdots \\ D_{b_7}\varphi \end{bmatrix} \equiv \begin{bmatrix} \varphi^0 \\ \varphi^1 \\ \vdots \\ \varphi^7 \end{bmatrix}$$

to be a vector-valued scaling function for this MRA.

Just by looking at Figure 1, observe that

$$R_0 = q^{-1}R_1 \cup \left[q^{-1}R_6 + \begin{pmatrix}1/2\\1/2\end{pmatrix}\right] = q^{-1}R_1 \cup q^{-1}\left(R_6 + \begin{pmatrix}0\\1\end{pmatrix}\right). \tag{2.1}$$

This equality implies $\chi_{R_0}(x) = \chi_{q^{-1}R_1}(x) + \chi_{q^{-1}\left(R_6+\binom{0}{1}\right)}(x)$, or, equivalently,

$$\varphi^0(x) = \varphi^1(qx) + \varphi^6\left(qx - \begin{pmatrix}0\\1\end{pmatrix}\right). \tag{2.2}$$

Applying D_{b_i} to the above equation (or just reading from Figure 1) we obtain

$$\left.\begin{array}{ll}
\varphi^0(x) = \varphi^1(qx) + \varphi^6\left(qx - \begin{pmatrix}0\\1\end{pmatrix}\right), & \varphi^1(x) = \varphi^2(qx) + \varphi^5\left(qx - \begin{pmatrix}0\\1\end{pmatrix}\right), \\
\varphi^2(x) = \varphi^3(qx) + \varphi^0\left(qx + \begin{pmatrix}1\\0\end{pmatrix}\right), & \varphi^3(x) = \varphi^4(qx) + \varphi^7\left(qx + \begin{pmatrix}1\\0\end{pmatrix}\right), \\
\varphi^4(x) = \varphi^5(qx) + \varphi^2\left(qx + \begin{pmatrix}0\\1\end{pmatrix}\right), & \varphi^5(x) = \varphi^6(qx) + \varphi^1\left(qx + \begin{pmatrix}0\\1\end{pmatrix}\right), \\
\varphi^6(x) = \varphi^7(qx) + \varphi^4\left(qx - \begin{pmatrix}1\\0\end{pmatrix}\right), & \varphi^7(x) = \varphi^0(qx) + \varphi^3\left(qx - \begin{pmatrix}1\\0\end{pmatrix}\right).
\end{array}\right\} \tag{2.3}$$

Let $\psi(x) = \varphi^1(qx) - \varphi^6\big(qx - \binom{0}{1}\big)$. It is easily seen that ψ is the desired Haar-like wavelet. Moreover, the system

$$\left\{D_{q^j} D_b T_k \psi : \ j \in \mathbb{Z},\ b \in B,\ k \in \mathbb{Z}^2\right\} \tag{2.4}$$

is an orthonormal basis for $L^2(\mathbb{R}^2)$. The function ψ is, indeed, the difference of two (appropriately normalized) characteristic functions of disjoint triangles. Our scaling function φ is, clearly, much simpler than the "twin dragon" that is associated with the classical quincunx affine system $\{D_{q^j}T_k : \ j \in \mathbb{Z},\ k \in \mathbb{Z}^2\}$ and the same is true for the corresponding wavelets.

Our wavelet system can also be expressed as a standard multi-wavelet associated with the quincunx affine system. Let us be more explicit about this point. It is clear that the sets $\{D_bT_k,\ b \in B,\ k \in \mathbb{Z}^2\}$ and $\{T_kD_b,\ b \in B,\ k \in \mathbb{Z}^2\}$ are identical since the matrices in B have integer entries and $|\det b| = 1$, $b \in B$. Thus,

$$\left\{D_{q^j} D_b T_k \psi : \ j \in \mathbb{Z},\ b \in B,\ k \in \mathbb{Z}^2\right\} = \left\{D_{q^j} T_k(D_b\psi) : \ j \in \mathbb{Z},\ b \in B,\ k \in \mathbb{Z}^2\right\}.$$

Let $\psi^i = D_{b_i}\psi$, $i = 0, 1, \ldots, 7$, and observe that (2.4) is obtained by applying the quincunx affine system to the eight generating functions $\psi^0, \psi^1, \ldots, \psi^7$. These eight generators form a Haar-like wavelet that is still considerably simpler than the one obtained from the "twin dragon" scaling function.

Let us end this section by writing down the "classical" equalities associated with the constructed MRA in the frequency domain (the two-scale equations, the Smith-Barnwell equality, etc., see [9, Ch. 2]). Let

$$\hat{\Phi} = \begin{bmatrix} \hat{\varphi}^0 \\ \hat{\varphi}^1 \\ \vdots \\ \hat{\varphi}^7 \end{bmatrix} \quad \text{and} \quad \hat{\Psi} = \begin{bmatrix} \hat{\psi}^0 \\ \hat{\psi}^1 \\ \vdots \\ \hat{\psi}^7 \end{bmatrix},$$

and the frequency variable ξ be represented by row vectors: $\xi = (\xi_1, \xi_2)$. Then $\xi q = (\xi_1 + \xi_2, \xi_1 - \xi_2)$ and Equations (2.3) imply

$$\hat{\Phi}(\xi q) = M_0(\xi)\hat{\Phi}(\xi) , \tag{2.5}$$

where the "low-pass filter" matrix M_0 is given by

$$M_0(\xi) = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & e(-\xi_2) & 0 \\ 0 & 0 & 1 & 0 & 0 & e(-\xi_2) & 0 & 0 \\ e(\xi_1) & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & e(\xi_1) \\ 0 & 0 & e(\xi_2) & 0 & 0 & 1 & 0 & 0 \\ 0 & e(\xi_2) & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e(-\xi_1) & 0 & 0 & 1 \\ 1 & 0 & 0 & e(-\xi_1) & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $e(\alpha) \equiv e^{2\pi i\alpha}$ for $\alpha \in \mathbb{R}$. Using $\beta = (\frac{1}{2}, \frac{1}{2})$, we define the "high-pass filter" matrix M_1 by

$$M_1(\xi) = M_0(\xi + \beta) . \tag{2.6}$$

Indeed, an elementary calculation shows that

$$M_0(\xi)M_0^*(\xi) + M_1(\xi)M_1^*(\xi) = I , \tag{2.7}$$

where M^* denotes the conjugate transpose of the matrix M. The last equality corresponds to the "classical" Smith-Barnwell equality. Finally, the wavelet Ψ satisfies

$$\hat{\Psi}(\xi q) = M_1(\xi)\hat{\Phi}(\xi) . \tag{2.8}$$

3. Some more examples and general observations

The Haar-like wavelet we constructed in the previous section is nonseparable; that is, it is not the Cartesian product of one-dimensional wavelets. There are many other examples of such Haar-like wavelets that enjoy similar properties and are associated with other finite groups B and composite dilation systems. Let us present the basic ideas of two more of these constructions. We will not develop them in a detailed way as we have done above; the reader, however, can easily provide similar formulae in these cases as well.

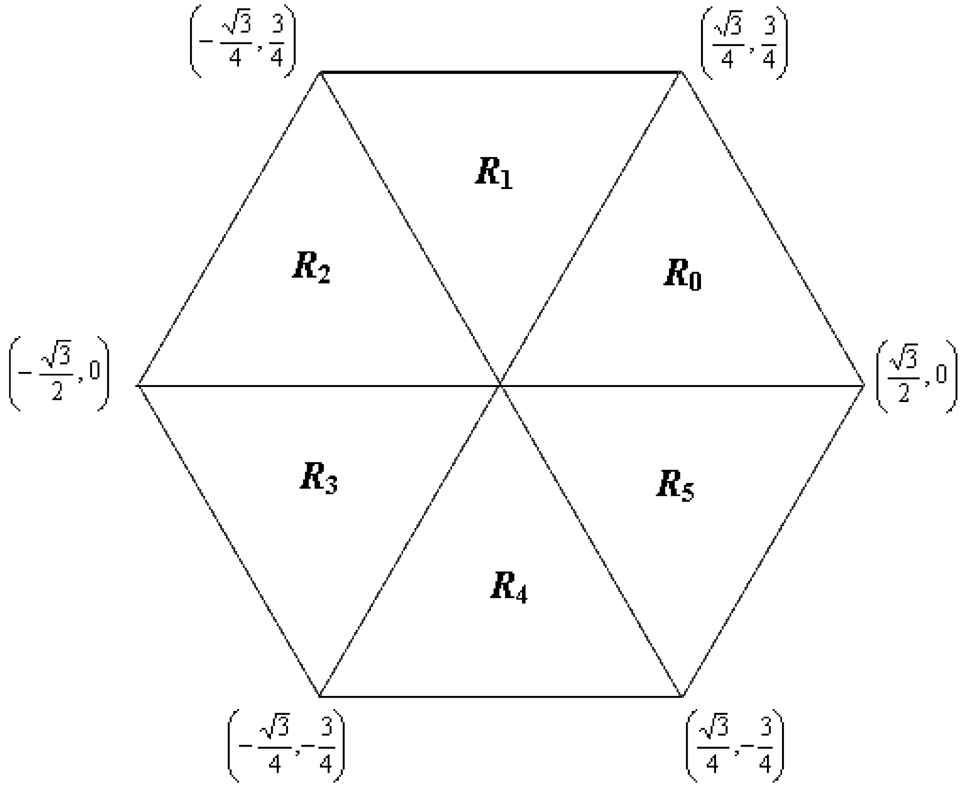

FIGURE 2 Scaling sets for the 6-element group.

Let B be the group of order 6 generated by the matrix $\rho = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$, which is the counter-clockwise rotation by $\pi/3$ radians. Consider the hexagon having the six vertices depicted in Figure 2. Let R_0 be the triangle with vertices $(0, 0)$, $(\sqrt{3}/2, 0)$, and $(\sqrt{3}/4, 3/4)$. The elements of B map R_0 onto the other triangles: $R_i = \rho^i R_0$, $i = 0, 1, 2, \ldots, 5$. Let $c = \frac{1}{4}\begin{pmatrix} 0 & 3\sqrt{3} \\ 6 & 3 \end{pmatrix}$ and $\Gamma = c\mathbb{Z}^2$. The space $V_0 \subset L^2(\mathbb{R}^2)$ in this example consists of those functions that are constant on Γ-translates of the triangles R_i, $i = 0, 1, 2, \ldots, 5$. The translates of the hexagon in Figure 2 by $\gamma \in \Gamma$ form a partition of $\mathbb{R}^2$ with the centers of the hexagons in the partition being the lattice points γ. Let $q = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$. The MRA we now consider is generated by the composite dilation system $\{D_{q^j} D_{\rho^i} T_\gamma : j \in \mathbb{Z},\ i = 0, 1, \ldots, 5,\ \gamma \in \Gamma\}$ applied to the scaling function $\varphi = \alpha \chi_{R_0}$ with α chosen so that $\|\varphi\|_2 = 1$. The spaces V_j are again q^{-j} dilates of V_0, that is, $V_j = D_{q^{-j}} V_0$, $j \in \mathbb{Z}$. In order to best describe the space V_1 we use Figure 3. In this picture we find the original hexagon H and, within H, the smaller hexagon $q^{-1}H$ which is the disjoint union of the triangles $I_i = \rho^i I_0$, $i = 0, 1, \ldots, 5$. The space V_1 is made up of all functions that are constant on $q^{-1}\Gamma$-translates of the triangles I_i, $i = 0, 1, 2, \ldots, 5$. Clearly, this implies $V_0 \subset V_1$. In this case, we can easily find three functions that will generate our Haar-like wavelet system: They are

$$\chi_{I_0} - \chi_{II_0} + \chi_{III_0} - \chi_{IV_0},\ \chi_{I_0} + \chi_{II_0} - \chi_{III_0} - \chi_{IV_0},\ \chi_{I_0} - \chi_{II_0} - \chi_{III_0} + \chi_{IV_0}, \tag{3.1}$$

which, after normalization in $L^2(\mathbb{R}^2)$, we denote by ψ^1, ψ^2, and ψ^3, respectively. These mutually orthogonal functions are also orthogonal to the scaling function $\varphi = \alpha \chi_{R_0} = \alpha(\chi_{I_0} + \chi_{II_0} + \chi_{III_0} + \chi_{IV_0})$. From these observations it follows that W_0, the orthogonal complement of V_0 in

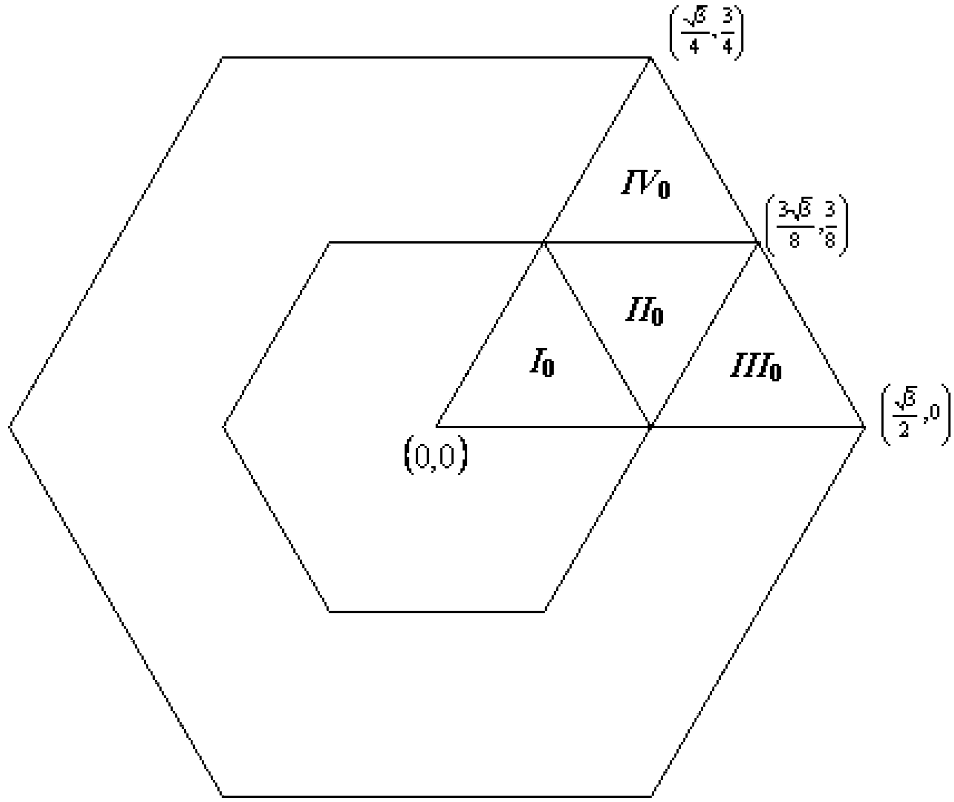

FIGURE 3 Fundamental wavelet regions for the 6-element group.

V_1, has an orthonormal basis $\{D_{\rho^i} T_\gamma \psi^\ell : j \in \mathbb{Z}, i = 0, 1, \ldots, 5, \gamma \in \Gamma, \ell = 1, 2, 3\}$. Since, as one can easily see, $c^{-1}Bc = B$, we have

$$\left\{D_{\rho^i} T_\gamma : i = 0, 1, \ldots, 5, \gamma \in \Gamma\right\} = \left\{T_\gamma D_{\rho^i} : i = 0, 1, \ldots, 5, \gamma \in \Gamma\right\}, \tag{3.2}$$

and, consequently,

$$\left\{D_{q^j} D_{\rho^i} T_\gamma \psi^\ell : j \in \mathbb{Z}, i = 0, 1, \ldots, 5, \gamma \in \Gamma, \ell = 1, 2, 3\right\} = \left\{D_{q^j} T_\gamma D_{\rho^i} \psi^\ell\right\} \tag{3.3}$$

is an orthonormal basis for $L^2(\mathbb{R}^2)$. The three generating functions are, indeed, Haar-like, as one can see from (3.1). The fact that there are three generators follows from $3 = |\det q| - 1$ (see [7]). As discussed in Section 2 and indicated in (3.3), we can regard this basis as generated by $3 \times 6 = 18$ functions $D_{\rho^i} \psi^\ell$, $i = 0, 1, \ldots, 5$, $\ell = 1, 2, 3$, via the application of the affine system $\{D_{q^j} T_\gamma : j \in \mathbb{Z}, \gamma \in \Gamma\}$. Another observation reveals that, instead of using the dilation q, one can use the dilation $a = 2I$ and obtain exactly the same basis. We used q since it compares well with the dilation in our other examples.

Our last example is also based on the hexagon, but now we consider as B its full group of symmetries generated by the matrices

$$r = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \quad \text{and} \quad \rho = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

Obviously, r represents the reflection (in $\mathbb{R}^2$) about the line through the origin with slope $1/\sqrt{3}$ and ρ is the rotation we used in the previous example. We can express elements $b_i \in B$, $i = 0, 1, \ldots, 11$, in the form $b_{2i} = \rho^i$, $i = 0, 1, \ldots, 5$, and $b_{2i+1} = \rho^i r$, $i = 0, 1, \ldots, 5$.

Consider the hexagon we introduced in Section 2 and divide it into 12 triangular regions R_i, so that $R_i = b_i R_0$, $i = 0, 1, \ldots, 11$, see Figure 4. The space V_0 that generates the MRA

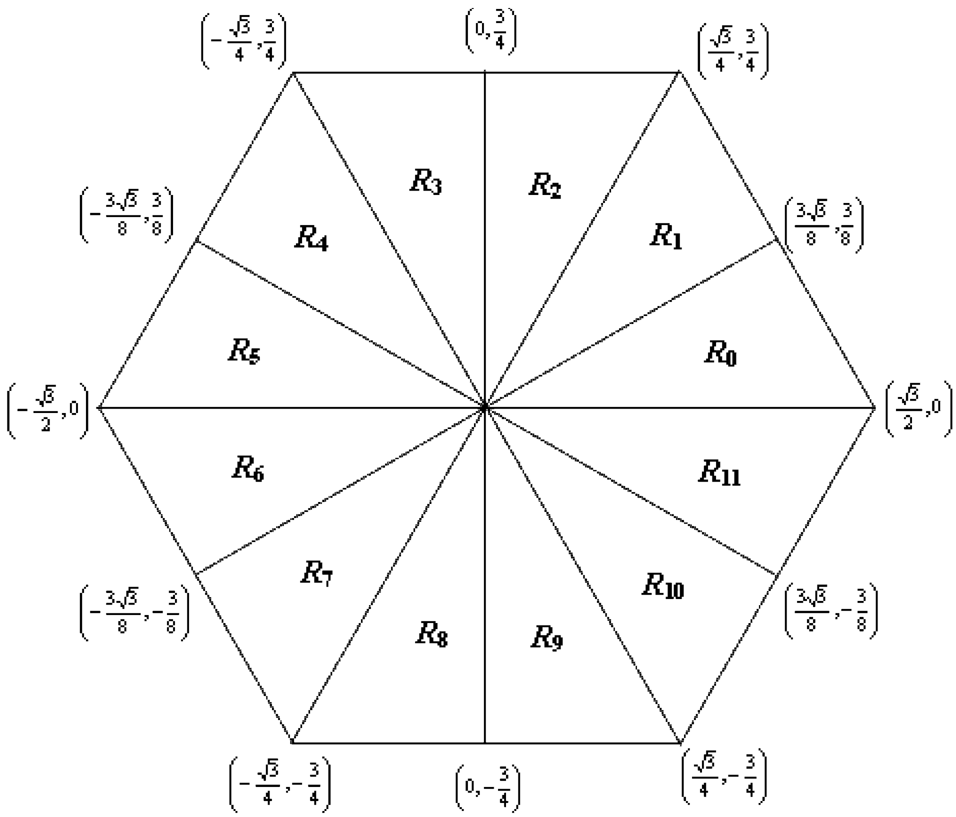

FIGURE 4 Scaling sets for the 12-element group.

in this example consists of all functions that are constant on the Γ-translates of the triangles R_i, $i = 0, 1, 2, \ldots, 11$. Let $q = \frac{1}{2}\begin{pmatrix} 3 & -\sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}$ be the rotation by $\pi/6$ radians multiplied by $\sqrt{3}$. The spaces V_j are, again, defined by the equality $V_j = D_{q^{-j}} V_0$, $j \in \mathbb{Z}$. Figure 5 presents the basic features of the space V_1. The function $\varphi = \alpha \chi_{R_0}$ with α chosen so that $\|\varphi\|_2 = 1$ is the scaling function that generates V_0. Indeed, $\{D_{b_i} T_\gamma \varphi : i = 0, 1, \ldots, 11, \gamma \in \Gamma\} = \{T_\gamma D_{b_i} \varphi : i = 0, 1, \ldots, 11, \gamma \in \Gamma\}$ is an orthonormal basis for V_0. The operator D_q maps the hexagon in Figure 4 onto the smaller hexagon with vertices $(\sqrt{3}/4, 1/4)$, $(0, 1/2)$, $(\sqrt{3}/4, -1/4)$, $\ldots$, which is depicted in Figure 5. We have also indicated the regions $q^{-1}R_i$, $i = 0, 1, 2, \ldots$. The regions I, II, and III are the analogs of the regions used in (3.1); the characteristic functions χ_I, χ_{II}, and χ_{III} can be used to express φ and define ψ^1 and ψ^2. This time there are two generating functions as $|\det q| - 1 = 2$. Explicitly, we let $\varphi = \alpha \chi_{R_0} = \alpha(\chi_I + \chi_{II} + \chi_{III})$, $\psi^1 = \beta(\chi_I - \chi_{II})$, and $\psi^2 = \gamma[\frac{1}{2}(\chi_I + \chi_{II}) - \chi_{III}]$ (where α, β, and γ are chosen so that $\|\varphi\|_2 = \|\psi^1\|_2 = \|\psi^2\|_2 = 1$). From this construction, we can easily see that the space W_0 generated by ψ^1, ψ^2, and their Γ-translates is the orthogonal complement of V_0 in V_1, which consists of the functions that are constant on $q^{-1}\Gamma$-translates of the sets $q^{-1}R_i$, $i = 0, 1, \ldots, 11$. Now one can easily verify the validity of all the properties of a composite dilation MRA and establish the analogs of the equalities (2.5)–(2.8).

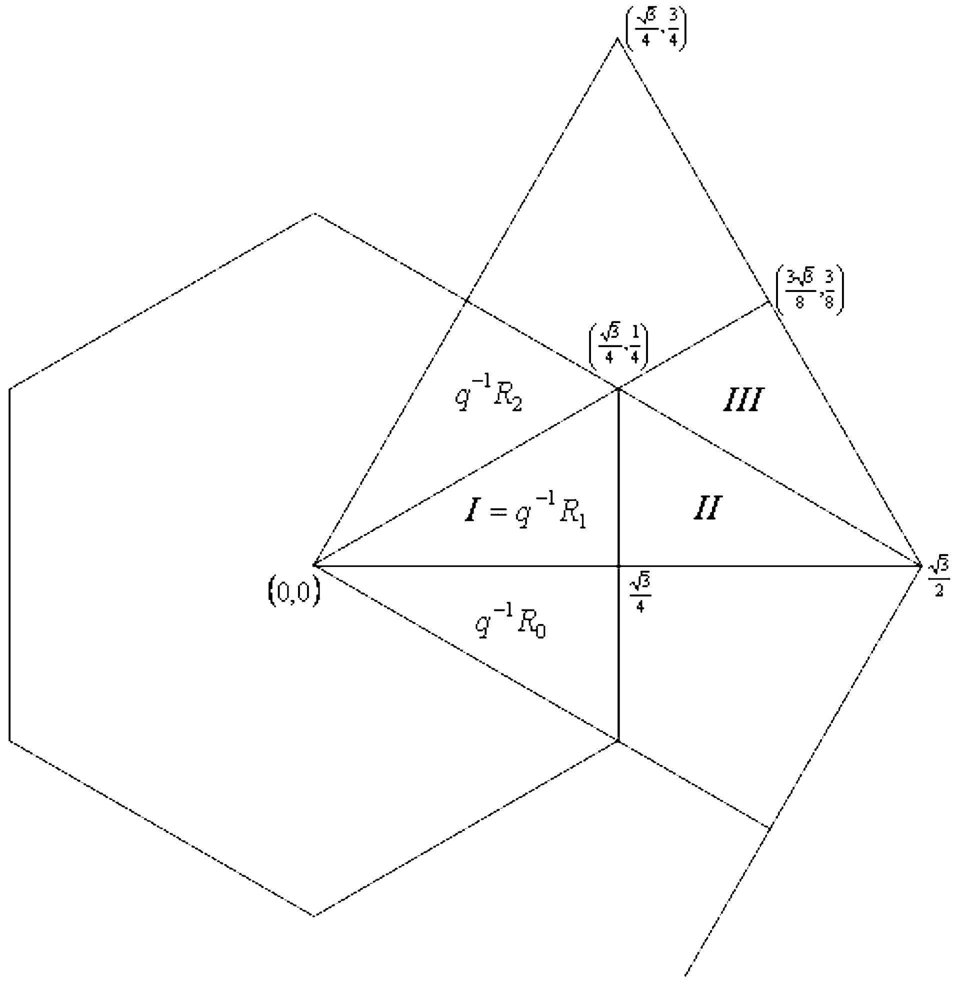

FIGURE 5 Fundamental wavelet regions for the 12-element group.

Other similar examples of composite dilation orthonormal wavelet bases have been developed (in higher dimensions and associated with various finite groups). Members of our research group are investigating in this area and are trying to classify those finite groups that are associated with Haar-like wavelets. A more challenging problem is to produce smoother compactly supported wavelets such as the ones obtained in [1]–[3]. Some preliminary results have been obtained by members of our group and some of the authors just cited.

We end this article by mentioning that in the original articles that introduced the composite dilation wavelets cited here we have given several examples of MSF (*Minimally Supported Frequency*) wavelets corresponding to various groups B. The absolute values of their Fourier transforms are characteristic functions of sets of measure 1. They provide several more examples of groups B whose elements are matrices with determinants equal to 1 in absolute value. In some instances it is not at all clear whether such composite dilation systems admit *any* compactly supported wavelets.

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