Wiener's Lemma and memory localization

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WAVELETS AND APPLICATIONS
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Acknowledgments

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Outline

Intro: Examples

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- Dual Frame Localization;
- Inverse Closedness of different classes of ΨDO;
- Memory localization of certain integral operators.

Goal: Indicate how different versions of Wiener's Lemma are responsible for each of these phenomena.

Reminder I

Theorem (Wiener, 1932)

If a periodic function f has an absolutely convergent Fourier series and never vanishes then the function $1/f$ also has an absolutely convergent Fourier series.

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If A is an invertible Laurent matrix with summable diagonals then the matrix A^{-1} also has summable diagonals.

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Wiener's Meta-Theorem

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Inverse operators usually possess the same or similar memory localization

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Proof idea.

If A is a localized operator consider a function $T(t)A$, where T is a group representation with respect to which the memory of A is defined. Localization is often equivalent to this function having a bounded holomorphic extension to a strip or an annulus in a complex plane. The latter property is preserved for the inverses. □

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- $T\colon \mathbb{R}^d \to End\mathcal{B}$ be an isometric representation of the group $\mathbb{R}^d$ with the following properties:
  - $T(\gamma)I = I$ for all $\gamma \in \mathbb{R}^d$;
  - $T(\gamma)(AB) = (T(\gamma)A)(T(\gamma)B)$ for all $\gamma \in \mathbb{R}^d, \; A, B \in \mathcal{B}$.

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- $A \in AP^T(\mathcal{B})$ if the function $\hat{A} : \mathbb{R}^d \to \mathcal{B}$, $\hat{A}(\gamma) = T(\gamma)A$, is continuous (in the topology of $\mathcal{B}$) and Bohr almost periodic.

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$$\hat{A}(\gamma) \sim \sum_{j\in\mathbb{R}^d} e^{2\pi i\langle\gamma,j\rangle} A_j, \;\; T(\gamma)A_j = e^{2\pi i\langle\gamma,j\rangle} A_j.$$

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$$A_j = \lim_{N\to\infty} \frac{1}{(2N)^d} \int_{[-N,N]^d} e^{-2\pi i\langle\gamma,j\rangle} T(-\gamma) A d\gamma.$$

Notation - continued

- A *weight* is a function $\nu : \mathbb{R}^d \to [1, \infty)$ such that $\nu(t+s) \le \nu(t)\nu(s),\ t, s \in \mathbb{R}^d$.

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- A weight is *admissible* if it satisfies the *GRS-condition*

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Definition

A *Baskakov algebra* $AP_\nu^T(\mathcal{B})$ is a subalgebra of $AP^T(\mathcal{B})$ containing all elements with the Fourier series summable with weight ν.

$$\|A\|_\nu = \sum_{j\in\mathbb{R}^d} \nu(j)\, \|A_j\| < \infty.$$

Almost periodic noncommutative Wiener's Lemma

Theorem (R. Balan, IK)

Let ν be an admissible weight. Then the subalgebra $AP_\nu^T(\mathcal{B}) \subset \mathcal{B}$ is inverse closed, that is, if $A \in AP_\nu^T(\mathcal{B})$ is invertible in $\mathcal{B}$ then $A^{-1} \in AP_\nu^T(\mathcal{B})$.

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Theorem (R. Balan, IK)

Let $\nu_\rho(j) = e^{\rho|j|}$, $j \in \mathbb{R}^d$, and $A \in AP_{\nu_\rho}^T(\mathcal{B})$ be invertible in $\mathcal{B}$. Then there exists $\bar{\rho} > 0$ such that $A^{-1} \in AP_{\nu_{\bar{\rho}}}^T(\mathcal{B})$.

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Frame Localization I

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Theorem (IK, K. Okoudjou)

If a Gabor frame generator belongs to a Wiener Amalgam space then the canonical dual generator and the associated Parseval frame generator do as well.

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If ν is an admissible weight, the canonical dual to a (T, ν)-localized frame is also (T, ν)-localized.

Localized Frames III

Theorem (A. Aldroubi, A. Baskakov, IK)

Under weak additional assumptions on ν any (T, ν)-localized frame is a Banach frame in L^q for all $q \in [1, \infty]$.

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Theorem (A. Aldroubi, A. Baskakov, IK)

A set of sampling for some $p \in [1, \infty]$ remains a set of sampling for all $q \in [1, \infty]$ if the sampling operator is sufficiently localized.

Inverse Closedness of different classes of ΨDO

$$(Ax)(t) = \int_{\mathbb{R}^2} \sigma(\frac{s+t}{2}, \xi) e^{2\pi i (t-s)\cdot\xi} x(s) ds d\xi;$$

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Theorem (IK, T. Strohmer)

If a ΨDO A in the Sjöstrand's class $\mathfrak{S}$ satisfies $\|A - I\|_{\mathfrak{S}} < 1$, then A admits canonical causal factorization within $\mathfrak{S}$. If in addition $A \in AP_1^T$, then it is enough to have $\|A - I\|_{\mathcal{B}} < 1$.

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Proof idea.

Differential operators are typically memoryless. If an integral operator is an inverse of a memoryless (differential) operator it should have exponential memory localization by Wiener's meta-theorem. □

References

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