Math 109 P-2 Instructor: Mihai Racovitan

Exam 2-A on sections 6.8 − 8.3 **Solutions** March 19

1. Solve the inequality, graph the solution and write the solution in interval notation.

$$-3(x+2) > -9$$

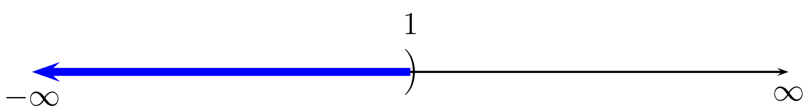
Solutions :

You can solve it in two ways. We will show one way.

$$-3(x+2) > -9 \Rightarrow -3x - 6 > -9 \Rightarrow \underbrace{-3x-6}_{+6} > \underbrace{-9}_{+6} \Rightarrow -3x > -3 \Big/ \div(-3) \Rightarrow$$

$$\Rightarrow \frac{-3x}{-3} < \frac{-3}{-3} \Rightarrow x < 1 \quad .$$

The graphic solution is



The interval solution is : $x \in (-\infty, 1)$. ■

2. Solve the inequality, graph the solution and write the solution in interval notation.

$$-4 \leq \frac{x+2}{3} \leq 4$$

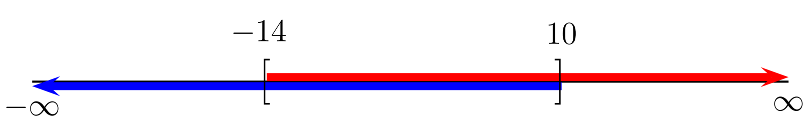
Solutions :

We do not need to separate into two inequalities, we can solve them together:

$$-4 \leq \frac{x+2}{3} \leq 4 \Big/ (\cdot 3) \Rightarrow -4\cdot 3 \leq \frac{x+2}{3}\cdot 3 \leq 4 \cdot 3 \Rightarrow -12 \leq x+2 \leq 12 \Rightarrow$$

$$\Rightarrow \underbrace{-12}_{-2} \leq \underbrace{x+2}_{-2} \leq \underbrace{12}_{-2} \Rightarrow -14 \leq x \leq 10 \quad .$$

The graphic solution is



The interval solution is : $x \in [-14, 10]$. ■

3. Solve. $\left|\frac{1}{2}\,x-3\right|-4=2$

<u>*Solutions*</u> :

Move first 4 on the right side, to have only the absolute value on the left side.

$$\left|\frac{1}{2}\,x-3\right|-4=2 \Rightarrow \left|\frac{1}{2}\,x-3\right|=6 \quad .$$

We have the following fact.

$$\boxed{\mid x \mid = k \ (k \geq 0) \quad \text{is equivalent to} \quad x=k \quad \text{or} \quad x=-k} \tag{1}$$

According to (1), we have that

$$\left|\frac{1}{2}\,x-3\right|=6 \quad \text{is equivalent to} \quad \frac{1}{2}\,x-3=6 \quad \text{or} \quad \frac{1}{2}\,x-3=-6$$

Solving the first equation, we have

$$\frac{1}{2}\,x-3=6 \Rightarrow \underbrace{\frac{1}{2}\,x-3}_{+3}=\underbrace{6}_{+3} \Rightarrow \frac{1}{2}\,x=9 \Big/ (\cdot 2) \Rightarrow x=18 \quad .$$

Solving the second equation, we have

$$\frac{1}{2}\,x-3=-6 \Rightarrow \underbrace{\frac{1}{2}\,x-3}_{+3}=\underbrace{-6}_{+3} \Rightarrow \frac{1}{2}\,x=-3 \Big/ (\cdot 2) \Rightarrow x=-6 \quad .$$

Therefore, we have two solutions : $\boxed{x=-6}$ and $\boxed{x=18}$. ■

4. Solve the inequality, graph the inequality, and write the solution set in interval notation.

$$\mid 2x+7 \mid < 13$$

<u>*Solutions*</u> :

We have the following fact.

$$\boxed{\mid x \mid < k \quad \text{is equivalent to} \quad -k < x < k \quad (k > 0)} \tag{2}$$

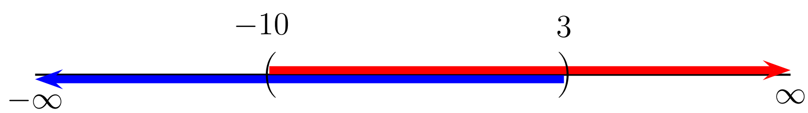
According to (2), we have

$$| 2x + 7 | < 13 \quad \text{is equivalent to} \quad -13 < 2x + 7 < 13$$

We do not need to separate into two inequalities, we can solve them together:

$$\underbrace{-13}_{-7} < \underbrace{2x + 7}_{-7} < \underbrace{13}_{-7} \Rightarrow -20 < 2x < 6 \Rightarrow \frac{-20}{2} < \frac{2x}{2} < \frac{6}{2} \Rightarrow -10 < x < 3 \quad .$$

The graphic solution is



The interval solution is : $x \in (-10, 3)$. ■

5. Solve the inequality, graph the inequality, and write the solution set in interval notation.

$$| 2x - 3 | \geq 3$$

<u>*Solutions*</u> :

We have the following fact. If $k \geq 0$, then

$$\boxed{| x | \geq k \quad \text{is equivalent to} \quad x \leq -k \quad \text{or} \quad x \geq k} \tag{3}$$

Therefore, we have, by (3), that

$$| 2x - 3 | \geq 3 \quad \text{is equivalent to} \quad 2x - 3 \leq -3 \quad \text{or} \quad 2x - 3 \geq 3$$

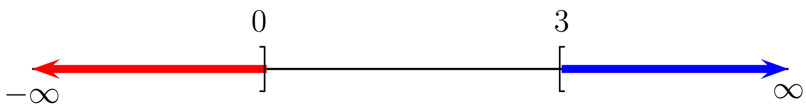
We cannot solve this set of inequalities together, we solve them separately. The first inequality is

$$2x - 3 \leq -3 \Rightarrow \underbrace{2x - 3}_{+3} \leq \underbrace{-3}_{+3} \Rightarrow 2x \leq 0 \Rightarrow \frac{2x}{2} \leq \frac{0}{2} \Rightarrow x \leq 0 \quad .$$

The second inequality is

$$2x - 3 \geq 3 \Rightarrow \underbrace{2x - 3}_{+3} \geq \underbrace{3}_{+3} \Rightarrow 2x \geq 6 \Rightarrow \frac{2x}{2} \geq \frac{6}{2} \Rightarrow x \geq 3 \quad .$$

The solution is given by $x \leq 0$ **OR** $x \geq 3$. Here is a graphic solution .



These interval solutions are considered together : $x \in (-\infty, 0] \cup [3, \infty)$. ■

6. Factor $x^2 + 4x + 4 - 9y^2$.

   Solutions :

   Group them like this:

$$x^2 + 4x + 4 - 9y^2 = (x^2 + 4x + 4) - 9y^2 \quad .$$

   The first is a perfect square: $x^2 + 4x + 4 = (x+2)^2$. Then,

$$x^2 + 4x + 4 - 9y^2 = (x^2 + 4x + 4) - 9y^2 = (x+2)^2 - 9y^2 = (x+2)^2 - (3y)^2 \quad .$$

   Use now the formula:

$$F^2 - L^2 = (F - L)(F + L) \quad . \tag{4}$$

   Then

$$x^2+4x+4-9y^2 = (\underbrace{x+2}_{F})^2 - (\underbrace{3y}_{L})^2 = [(x+2) - 3y]\,[(x+2) + 3y] = (x+2-3y)(x+2+3y) \ . \ ■$$

7. Factor $x^3 + 64y^3$.

   Solutions :

   We have here the sum of two cubes: x^3 and $64y^3 = (4y)^3$. The sum of two cubes factor as

$$a^3 + b^3 = (a+b)(a^2 + ab + b^2) \quad . \tag{5}$$

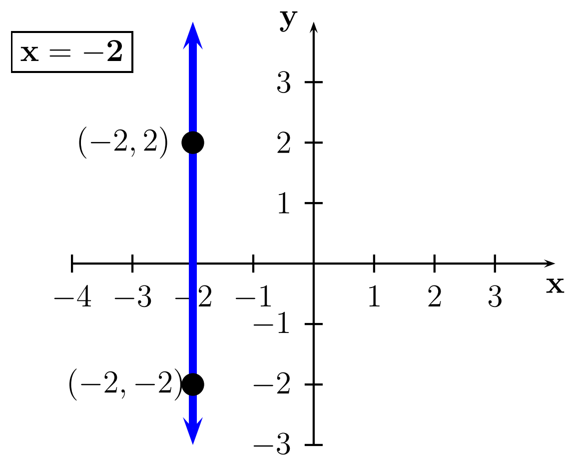
   Therefore

$$x^3 + 64y^3 = x^3 + (4y)^3 = (x+4y)\left[x^2 + x \cdot (4y) + (4y)^2\right] = (x+4y)(x^2 + 4xy + 16y^2) \quad . \ ■$$

8. Graph on the rectangular coordinate system $x = -2$.

   Solutions :

   This is a vertical line (y varies, but $x = -2$). You can choose two points that have the x-coordinate equal to -2; on the graph, $(-2, -2)$ and $(-2, 2)$.



9. Use synthetic division to do the division.

   $(2x^3+4x^2+3x-1)\div(x+2)$ **Remainder:**________ **, Quotient:**__________________

   <u>*Solutions*</u> :

   We use the **synthetic division**. The coefficients of $2x^3+4x^2+3x-1$ are 2 for x^3, 4 for x^2, 3 for x, and -1 for the free term. We divide by $x+2$, so we use $r=-2$:

	x^3		x^2		x		$free$
-2 |	2		4		3		-1
		$\nearrow$	-4	$\nearrow$	0	$\nearrow$	-6
	2		0		3		$\boxed{-7}$

   So, the remainder is -7 .

   The rest of them are the coefficients of the quotient: 2 for x^2, 0 for x and 3 for the free term. The quotient is $2x^2+0x+3=2x^2+3$. ■

10. Find the slope of the line that passes through the points: $(2,-4)$ and $(6,4)$.

    <u>*Solutions*</u> :

    The slope formula, if the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ are given, is:

$$m=\frac{y_2-y_1}{x_2-x_1} \quad . \tag{6}$$

    Then, for our problem - considering $P(x_1,y_1)=(2,-4)$ and $Q(x_2,y_2)=(6,4)$, according to (6) we have that

$$m=\frac{4-(-4)}{6-2}=\frac{4+4}{6-2}=\frac{8}{4}=2 \quad . \quad ■$$

11. Find the slope of the line $y = -4$.

<u>*Solutions*</u> :

You should know by now that vertical lines have undefined slope, and horizontal lines have slope 0. $y = -4$ is a horizontal line, so it has slope zero. However, if you did not know that, then let us choose any two points; y is always -4:

x	y
0	-4
2	-4

Then, by using the formula (6) for the points $P(x_1, y_1) = (0, -4)$ and $Q(x_2, y_2) = (2, -4)$:

$$m = \frac{-4-(-4)}{2-0} = \frac{-4+4}{2-0} = \frac{0}{2} = 0 \quad . \quad \blacksquare$$

12. Find the slope of the line that is perpendicular to the graph of $y = -\frac{3}{4}\,x - \frac{1}{4}$.

<u>*Solutions*</u> :

Two lines of slopes m_1 and m_2, respectively, are perpendicular on each other if

$$m_1 \cdot m_2 = -1 \quad , \text{or} \quad m_2 = -\frac{1}{m_1} \quad . \tag{7}$$

The slope of the line $y = -\frac{3}{4}\,x - \frac{1}{4}$ is $m_1 = -\frac{3}{4}$ (the line is in the slope-intercept form $y = mx + b$) .

Then, the slope of the line that is perpendicular to the graph of $y = -\frac{3}{4}\,x - \frac{1}{4}$ is, according to second relation in (7):

$$m_2 = -\frac{1}{m_1} = -\frac{1}{-\frac{3}{4}} = \frac{1}{\frac{3}{4}} = 1 \cdot \frac{4}{3} = \frac{4}{3} \quad . \quad \blacksquare$$

13. Determine whether the two lines $4x+8y = 10$ and $2x = 12-4y$ are perpendicular, parallel, or neither.

<u>*Solutions*</u> :

We need to check if their slopes satisfy (7), or if the slopes are equal. To find their slopes, we reduce their equations to the slope-intercept form $y = mx + b$:

$$4x + 8y = 10 \Rightarrow \underbrace{4x + 8y}_{-4x} = \underbrace{10}_{-4x} \Rightarrow 8y = -4x + 10 \Rightarrow \frac{8y}{8} = \frac{-4x}{8} + \frac{10}{8} \Rightarrow y = -\frac{1}{2}\,x + \frac{10}{8}$$

so, its slope is $m_1 = -\dfrac{1}{2}$.

$$2x = 12 - 4y \Rightarrow \underbrace{2x}_{+4y} = \underbrace{12 - 4y}_{+4y} \Rightarrow 4y + 2x = 12 \Rightarrow \underbrace{4y + 2x}_{-2x} = \underbrace{12}_{-2x} \Rightarrow$$

$$\Rightarrow 4y = -2x + 12 \Rightarrow \frac{4y}{4} = \frac{-2x}{4} + \frac{12}{4} \Rightarrow y = -\frac{1}{2} \ + 3$$

so, its slope is $m_2 = -\dfrac{1}{2}$.

Since $m_1 = m_2$, the two lines $4x + 8y = 10$ and $2x = 12 - 4y$ are parallel. ■

14. Find the equation of the line that passes through the point $(2, 3)$ with a slope of -2.

<u>*Solutions*</u> :

We have a point $P(x_1, y_1)$ and the slope m, so we can use the point-slope formula:

$$y - y_1 = m(x - x_1) \quad . \tag{8}$$

In our case, $P(x_1, y_1) = (2, 3)$ and $m = -2$, so (after simplification)

$$y - 3 = -2(x - 2) \Rightarrow y - 3 = -2x + 4 \Rightarrow y = 3 - 2x + 4 \Rightarrow y = -2x + 7 \quad . \quad ■$$

15. Solve : $\mid 4x - 3 \mid = \mid 2x + 5 \mid$.

<u>*Solutions*</u> :

We have the following equivalence:

$$\mid x \mid = \mid y \mid \ \text{ is equivalent } x = y \text{ or } x = -y \quad .$$

Then

$$\mid 4x - 3 \mid = \mid 2x + 5 \mid \ \text{ is equivalent } 4x - 3 = 2x + 5 \text{ or } 4x - 3 = -(2x + 5) \quad .$$

The first equality is

$$4x - 3 = 2x + 5 \Rightarrow \underbrace{4x - 3}_{+3} = \underbrace{2x + 5}_{+3} \Rightarrow 4x = 2x + 8 \Rightarrow \underbrace{4x}_{-2x} = \underbrace{2x + 8}_{-2x} \Rightarrow$$

$$\Rightarrow 2x = 8 \Rightarrow \frac{2x}{2} = \frac{8}{2} \Rightarrow x = 4 \quad .$$

The second equality is

$$4x - 3 = -(2x + 5) \Rightarrow 4x - 3 = -2x - 5 \Rightarrow \underbrace{4x - 3}_{+3} = \underbrace{-2x - 5}_{+3} \Rightarrow 4x = -2x - 2 \Rightarrow$$

$$\Rightarrow \underbrace{4x}_{+2x} = \underbrace{-2x-2}_{+2x} \Rightarrow 6x = -2 \Rightarrow \frac{6x}{6} = \frac{-2}{6} \Rightarrow x = -\frac{1}{3} \quad .$$

We have two solutions $x = 4$ and $x = -\dfrac{1}{3}$. ■

16. Find the equation of the line that passes through the point $(-6, -1)$ perpendicular to $2x + y = 8$.

<u>*Solutions*</u> :

Two lines are perpendicular if their slopes satisfy (7). The given line $2x + y = 8$ has the slope m_1 :

$$2x + y = 8 \Rightarrow \underbrace{2x+y}_{-2x} = \underbrace{8}_{-2x} \Rightarrow y = -2x + 8 \Rightarrow m_1 = -2 \quad .$$

Then, according to (7), the line perpendicular on $2x + y = 8$ has the slope:

$$m_2 = -\frac{1}{m_1} = -\frac{1}{-2} = \frac{1}{2} \quad .$$

Then the line passes through the point $(-6, -1)$ and has the slope $m_2 = \dfrac{1}{2}$ is

$$y - y_1 = m(x - x_1) \Rightarrow y - (-1) = \frac{1}{2}\,(x - (-6)) \Rightarrow y + 1 = \frac{1}{2}\,(x + 6) \quad . \quad ■$$

17. (a) Rewrite the equation of $2x + 3y = 6$ in slope-intercept form .

    (b) Find the slope and the y-intercept of the equation $2x + 3y = 6$.

    (c) Graph $2x + 3y = 6$.

<u>*Solutions*</u> :

(a) The equation of $2x + 3y = 6$ in slope-intercept form is

$$2x + 3y = 6 \Rightarrow \underbrace{2x+3y}_{-2x} = \underbrace{6}_{-2x} \Rightarrow 3y = -2x + 6 \Rightarrow \frac{3y}{3} = \frac{-2x}{3} + \frac{6}{3} \Rightarrow y = -\frac{2}{3}\,x + 2$$

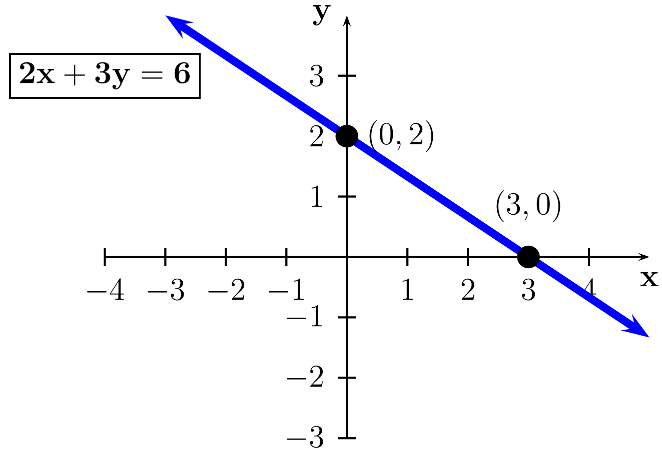
(b) The slope and the y-intercept of the equation $2x + 3y = 6$ can be read from (a):

$$m = -\frac{2}{3} \quad , \quad b = 2 \text{ , so } (0, 2) \quad .$$

(c) Graph $2x+3y=6$. We have already the y-intercept : $(0,2)$. The x-intercept is

$$y=0 \text{ in } 2x+3y=6 \Rightarrow 2x+3\cdot 0=6 \Rightarrow 2x=6 \Rightarrow x=3 \Rightarrow (3,0)$$

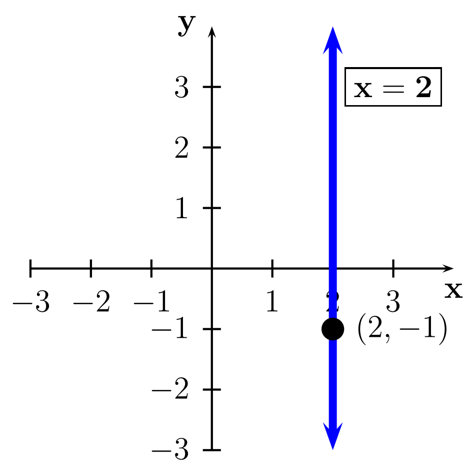
Then



18. Find the equation of the vertical line that passes through the point $(2,-1)$.

    Solutions :

    The vertical line that passes through the point $(2,-1)$ is $x=2$, i.e., exactly the x-coordinate of the given point.



19. Solve the inequality : $|\,2x-1\,|+2<2$.

    Solutions :

    Reduce $|\,2x-1\,|+2<2$ to an inequality that has the absolute value only on the left side:

$$|\,2x-1\,|+2<2 \Rightarrow \underbrace{|\,2x-1\,|+2}_{-2} < \underbrace{2}_{-2} \Rightarrow |\,2x-1\,| < 0$$

The last inequality is a contradiction because $| 2x - 1 |$ is always positive, so it cannot be less than 0. Therefore, the inequality has no solutions . ■

20. Use the **Factor Theorem** and determine whether the first expression is a factor of $P(x)$ or not.

$$x + 1 \quad ; \quad P(x) = x^3 + 2x^2 - 2x - 3$$

<u>*Solutions*</u> :

The **Factor Theorem** states that $x - r$ is a factor of the polynomial $P(x)$ **if and only if** $P(r) = 0$.

$r = -1$ in our case, because $x + 1 = x - (-1)$. Then,

$$P(-1) = (-1)^3 + 2(-1)^2 - 2 \cdot (-1) - 3 = -1 + 2 + 2 - 3 = 0 \quad .$$

Then, according to the Factor Theorem, $x + 1$ is a factor of $P(x) = x^3 + 2x^2 - 2x - 3$, because $P(-1) = 0$. ■