Covering Sections 3.2 – 3.6, 4.1 – 4.3

Section 3.2 The Graph of a Function

This includes notes on 3.1, which you should review.

A function from a set X to a set Y is a rule or correspondence that associates with each element of X exactly one element of Y. (In Math 110, these sets usually consist of real numbers.) With the function notation $y = f(x)$, each x value has only one corresponding y value. For the graph of the function, this means that no vertical line can intersect the graph in more than one point (see the vertical line test on page 232).

You can think of a function as being like a program. The x-values are the inputs, and the y-values are the outputs. The possible inputs form the domain of the function, and the possible outputs form its range. For the functions that we are dealing with, the numbers that we need to exclude from the domain are numbers that lead to division by zero, or the square root of a negative number. (If the function was a program, trying to divide by zero or take the square root of a negative number would give an error message.)

Given two functions, we can make a new function from their sum, difference, product, or quotient. (See pages 226–227.) Sample questions: p 336 #9, 25, 27, 29

Section 3.3 Properties of Functions

A function is called even if $f(-x) = f(x)$ (the graph is symmetric about the y-axis) and odd if $f(-x) = -f(x)$ (the graph that is symmetric about the origin).

A function is increasing on an interval if its values keep going up, and decreasing on an interval if its values keep going down. A high point on the graph is called a local maximum, and this corresponds to a change from increasing to decreasing. A low point on the graph is called a local minimum, and corresponds to a change from decreasing to increasing. *Note: Since we are not using calculators, you won't be asked to actually compute local maximum and local minimum values in this section.*

The average rate of change of a function is found by dividing the change in y by the change in x. If you go from x to c on the x-axis, then the corresponding change in y is $f(x) - f(c)$. We get this formula for the average rate of change: $\dfrac{f(x) - f(c)}{x - c}$, where $x \neq c$. Sample questions: p 249 # 43, 49, 47, 53, 67.

Section 3.4 A Library of Functions

This section gives the graphs of some functions you need to be able to recognize and to graph on your own.

Straight lines: $f(x) = mx + b$ The square function: $f(x) = x^2$ The cube function: $f(x) = x^3$

The square root function: $f(x) = \sqrt{x}$ The cube root function: $f(x) = \sqrt[3]{x}$

The reciprocal function: $f(x) = \dfrac{1}{x}$ The absolute value function: $f(x) = |x|$

There is no reason that a function has to have the same formula at each point in its domain. Of course, which formula is used for which numbers has to be spelled out very carefully. These functions get their name from being defined in "pieces". Example: the absolute value function $f(x) = |x|$ takes any number and makes it non-negative. It is convenient to express this with two different formulas: if x is already positive or zero, we don't need to make any change. But if x is negative, we need to change the sign.

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Sample problems: p 259 # 19, 21, 25, 31, 37.

Section 3.5 Graphing Techniques: Transformations

The basic model for a linear function is $f(x) = x$, whose graph is a straight line through the origin that slopes up at a 45^o angle. The family of linear functions includes all functions of the form $f(x) = ax + b$. We can get all of these by multiplying the basic example by a and adding b. The numbers a and b tell us all about the new line: if a is positive it slopes up, if a is negative it slopes down; b gives the y-intercept, and tells how far the line has been moved up (if $b > 0$) or down (if $b < 0$).

In Section 3.4, besides linear functions, we studied the basic examples of several families of functions:

quadratic functions	$f(x) = x^2$	cubic functions	$f(x) = x^3$
square root functions	$f(x) = \sqrt{x}$	cube root functions	$f(x) = \sqrt[3]{x}$
reciprocal functions	$f(x) = \dfrac{1}{x}$	absolute value functions	$f(x) =\mid x \mid$

In Section 3.5 we study the graphs that we get when the basic examples in each family are shifted up or down, shifted left or right, stretched or compressed, or reflected about one of the axes. We start with a function $f(x)$, and positive numbers a, h, and k, where $a > 1$. Changing the function does the following:

$f(x)+k$	shift up by k	$f(x)-k$	shift down by k
$f(x-h)$	shift right by h	$f(x+h)$	shift left by h
$af(x)$	stretch vertically	$\frac{1}{a}f(x)$	compress vertically
$-f(x)$	reflect about the y-axis	$f(-x)$	reflect about the x-axis

Sample questions: p272 #27,32,53,59,69

Section 3.6 Mathematical Models

Some of the models involve geometry. You should review the Pythagorean theorem on page 30, and the geometry formulas on page 31 (for the area of a rectangle, a triangle, or a circle). You need to know that the volume of a box is its length times width times height. On the exam, if you need any other formulas for volumes, the question will include the formula.

There are also some basic models related to economics. You need to remember that when items are sold the revenue (money taken in) is found by multiplying the number of items sold by the price per unit. That's just common sense, and you shouldn't have to make a big deal about remembering the formula $R(x) = px$, where x is the number sold and p is the price per unit.

Some of the problems in the text ask you to find the model and then find a maximum or minimum value, using a calculator. Obviously, though this is an important idea, we will not test you on it. But if it happens that the model gives you a quadratic function, then it *is* a fair question to ask you to find the maximum or minimum value on the exam, because you can use the techniques from Section 4.1. Sample questions: p280 #5,8,15,31

Section 4.1 Quadratic Functions and Models

The general form of a quadratic function is $f(x) = a(x-h)^2 + k$, where (h, k) is the vertex of the graph (which is a parabola). You can see from the formula that h gives the left/right shift while k gives the up/down shift. The coefficient a represents a vertical stretch or compression. Since the basic member of this family is $f(x) = x^2$, whose graph opens up, the graph of $f(x) = a(x-h)^2 + k$ will open up if a is positive, and down if a is negative. If the graph opens up, its height is minimum at the vertex; if the graph opens down, its height is maximum at the vertex.

If a quadratic is given in the form $f(x) = ax^2 + bx + c$, then the x-coordinate of its vertex is $x = -\frac{b}{2a}$. Since you already know the quadratic formula, you can remember it as part of the formula:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} .$$

This way of looking at the quadratic formula shows that if the graph has x-intercepts, they occur as points on either side of the line $x = -\frac{b}{2a}$, which is the axis of symmetry of the graph.

The models fall into some general categories: questions involving geometry formulas, questions on demand equations and revenue (provided the demand equation is a linear equation), and physics models of the motion of a projectile. If there is a question about motion, you would be given the equation. Of course, you need to know the same formulas as in Section 3.6. Each problem is different, so the best advice is to just use your common sense in setting up the model. If you have trouble finding a formula, try working with some numbers in some simple cases.

Sample questions: p306 #13,15,27,31,37,43,61,73,79,83

Section 4.2 Polynomial Functions

The general form of a polynomial is $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. The degree of $f(x)$ is the largest exponent in the formula. Linear functions $f(x) = mx+b$ and quadratic functions $f(x) = ax^2+bx+c$ are the simplest cases. If $|x|$ is large, then the term $a_n x^n$ is much larger than the others, so the "big picture" of $f(x)$ is that its graph follows the pattern of x^n, flipped over if a_n is negative.

The number of different x-intercepts of a polynomial of degree n is at most n, because a polynomial equation of degree n has at most n roots. The same is true of any horizontal line–the graph of a polynomial of degree n can cross the line at most n times. This means that the graph has at most $n-1$ "turning points" (see the discussion on the top of page 321), and this helps you in graphing.

Finally, a polynomial has two types of behavior at an x-intercept. It may cross the x-axis, like $y = x^3$, or it may just touch the x-axis, like $y = x^2$. If you can factor the function completely, you can tell whether it crosses or touches by looking at the exponent of the factor that corresponds to the root you are interested in. If the exponent is odd, the graph will cross the axis because the y-values will change sign, but if the exponent is even, the graph will just touch the axis and stay on the same side.

Review the summaries on pages 322 and 326. Sample questions: p327 #31,37,45,55,63,67,83

Section 4.3 Rational Functions

We have been building up to more and more complicated functions. This section deals with some basic properties of functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. These are called *rational* functions. The functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$ are two familiar examples.

Just as $f(x) = \frac{1}{x}$ has a graph that is asymptotic to the axes, a general rational function can have horizontal and vertical asymptotes. It may or may not cross the x-axis.

x-intercepts: The only way $f(x)$ can be zero is if the numerator is zero, so you can find the x-intercepts by setting the numerator equal to zero, and solving the equation $p(x) = 0$.

vertical asymptotes: These can be found by looking at the values of x at which $f(x)$ is not defined (because of division by zero). You just need to set the denominator equal to zero, and solve the equation $q(x) = 0$. Note: you must first make sure that the numerator and denominator do not have any common factors.

horizontal asymptotes: if $|x|$ is large, the function $f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + a_1 x + a_0}$ behaves like $y = \frac{a_m x^m}{b_n x^n}$, just like we learned for polynomials. If the numerator and denominator have the same degree, this reduces to a constant, and gives the equation of the asymptote. If the denominator has larger degree than the numerator, then $y = 0$ is a horizontal asymptote. If the numerator has larger degree than the denominator, then there is no horizontal asymptote (you will not be tested on oblique asymptotes). To just find the horizontal asymptotes you do *not* need to use long division to write $f(x)$ as a "mixed fraction."

Sample questions: p339 #15,21,27,35,39,41,45,47