

2. Let A be a 5×3 matrix and let B be a 3×5 matrix.

Exactly one of the following statements is always true.

Which statement is always true ?

(a) $AB = BA$

(b) $A^T A$ is 3×3 and $B^T A$ is 5×3

(c) $A^T B^T$ is 3×3 and BB^T is 5×5

(d) AB is 5×5 and BA is undefined

(e) AB is 5×5 and BA is 3×3

Solution :

Since A is a 5×3 matrix and B be a 3×5 matrix, then A^T is a 3×5 matrix, B^T is a 5×3 matrix, AB is a 5×5 matrix, BA is a 3×3 matrix, $A^T A$ is a 3×3 matrix, BB^T is a 3×3 matrix, $B^T A$ is an undefined matrix, and $A^T B^T$ is a 3×3 matrix.

The correct answer is (e) . ■

3. Suppose that $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Exactly one of the following statements is true.

Which statement is true ?

(a) Only C is invertible.

(b) Only D is invertible.

(c) Only E is invertible.

(d) Only C and E are invertible.

(e) Only D and E are invertible.

Solution :

Recall that a matrix is invertible if its determinant is non-zero (all **Identity Matrices** have a non-zero determinant, for example).

We have four 2×2 matrices, A , B , C , and D . Recall that the determinant of a 2×2 matrix

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be computed by the formula $\det(M) = a \cdot d - b \cdot c$. Therefore,

$$\det(A) = 1 \cdot 0 - 0 \cdot 0 = 0 , \det(B) = 1 \cdot 0 - 1 \cdot 0 = 0 , \det(C) = 1 \cdot 1 - 1 \cdot 0 = 1 , \text{ and}$$

$$\det(D) = 1 \cdot 1 - 1 \cdot 1 = 0 \Rightarrow \text{ only } C \text{ is invertible .}$$

About E , this is a 3×3 matrix, and we cannot use the same technique for computing its determinant. But E is the 3×3 **Identity Matrix** so it is invertible also.

The correct answer is (d) . ■

4. A line L passes through the point $(-4, 5)$ and is perpendicular to $y = -\frac{1}{2} \cdot x + 3$.

The equation representing L is given by

- (a) $y = \frac{1}{2}x + 7$ (c) $y = -\frac{1}{2}x + 3$ (e) none of the above.
(b) $y = -2x - 3$ (d) $y = 2x + 13$

Solution :

The slope-intercept equation of L is $y = m \cdot x + b$, where m is the slope and b the y -intercept.

The slope of the line $y = -\frac{1}{2} \cdot x + 3$ is $m_1 = -\frac{1}{2}$. The line L is perpendicular on the line $y = -\frac{1}{2} \cdot x + 3$ if

$$m = -\frac{1}{m_1} = -\frac{1}{-\frac{1}{2}} = 2.$$

We know by now that $L : y = 2x + b$. To find b , we know that the point $(-4, 5)$ is on our line. Therefore,

$$(-4, 5) \Rightarrow 5 = 2 \cdot (-4) + b \Rightarrow b = 13.$$

Therefore, $L : y = 2x + 13$.

The correct answer is (d). ■

5. Suppose that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

Exactly one of the following statements is true.

Which statement is true ?

- (a) $AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $AB \neq BA$

- (c) A is invertible and its inverse is B (d) A is invertible and its inverse is $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$

- (e) None of the above.

Solution :

A is invertible since has a non-zero determinant, $\det(A) = 1 \cdot 4 - 2 \cdot 3 = -2 \neq 0$.

To check that if the matrix B is its inverse, we have to multiply them, and to see if we get the identity matrix:

$$AB = BA = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \cdot (-2) + 2 \cdot \frac{3}{2} & 1 \cdot 1 + 2 \cdot \left(-\frac{1}{2}\right) \\ 3 \cdot (-2) + 4 \cdot \frac{3}{2} & 3 \cdot 1 + 4 \cdot \left(-\frac{1}{2}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

You have to check also that $BA = I$; but that it is similar to the above computation.

Now, (a) is false, by the above computation. Also, (b) is false since they are equal. Finally, (d) is false since is not the right inverse.

The correct answer is (c) . ■

6. Suppose that $A = \begin{bmatrix} 2 & 1 \\ 7 & 3 \end{bmatrix}$.

The inverse A^{-1} is given by

(a) $\begin{bmatrix} 3 & -1 \\ -7 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$ (e) none of the above.

(b) $\begin{bmatrix} -3 & 1 \\ 7 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & -1 \\ -7 & -2 \end{bmatrix}$

Solution :

For a 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the existence of its inverse is conditioned by the fact that its determinant $\det(M) = a \cdot d - b \cdot c \neq 0$. Then, its inverse is

$$M^{-1} = \frac{1}{\det(M)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

In our case, $\det(A) = 2 \cdot 3 - 1 \cdot 7 = -1 \neq 0$. Therefore, its inverse is

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} 3 & -1 \\ -7 & 2 \end{bmatrix} = \frac{1}{-1} \cdot \begin{bmatrix} 3 & -1 \\ -7 & 2 \end{bmatrix} = - \begin{bmatrix} 3 & -1 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 7 & -2 \end{bmatrix}.$$

The correct answer is (b) . ■

7. The system

$$\begin{cases} 2x + 4y + 6z = 3 \\ -x + 3y + z = 2 \\ x + 7y + 7z = 1 \end{cases}$$

has

- (a) no solutions. (b) a unique solution with $x = 1$.
(c) a unique solution with $z = 2$. (d) a unique solution with $y = -1$.
(e) an infinite number of solutions.

Solution :

We can use the Gauss-Jordan elimination method; first, add the second equation to the first:

$$\begin{cases} 2x + 4y + 6z = 3 \\ -x + 3y + z = 2 \\ x + 7y + 7z = 1 \end{cases} \Rightarrow \begin{cases} x + 7y + 7z = 5 \\ -x + 3y + z = 2 \\ x + 7y + 7z = 1 \end{cases}$$

Now compare the first and the last equations: $x + 7y + 7z = 5$ and $x + 7y + 7z = 1$. This leads to an absurdity: $5 = 1$. So we have no solutions for our system since two of its equations are incompatible.

The correct answer is (a) . ■

Second solution :

Form the augmented matrix:

$$\begin{aligned} \left[\begin{array}{ccc|c} 2 & 4 & 6 & 3 \\ -1 & 3 & 1 & 2 \\ 1 & 7 & 7 & 1 \end{array} \right] & \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 7 & 7 & 1 \\ -1 & 3 & 1 & 2 \\ 2 & 4 & 6 & 3 \end{array} \right] & \xrightarrow{\begin{array}{l} R_2 + R_1 \\ R_3 - 2 \cdot R_1 \end{array}} \\ \left[\begin{array}{ccc|c} 1 & 7 & 7 & 1 \\ 0 & 10 & 8 & 3 \\ 0 & -10 & -8 & 1 \end{array} \right] & \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 7 & 7 & 1 \\ 0 & 10 & 8 & 3 \\ 0 & 0 & 0 & 4 \end{array} \right] \end{aligned}$$

Last row has zero everywhere except the free term, which is 4. This implies that $0 \cdot x + 0 \cdot y + 0 \cdot z = 4 \Rightarrow 0 = 4$, absurd.

The correct answer is (a) . ■

8. The system

$$\begin{cases} x + 2y + 3z = 1 \\ -x + 3y + 2z = 4 \\ x + 7y + 13z = 6 \end{cases}$$

has

(a) a unique solution with $y = 1$. (b) a unique solution with $y = -2$.

(c) a unique solution with $y = 4$. (d) no solutions.

(e) an infinite number of solutions.

Solution :

We can use the Gauss-Jordan elimination method; first, add the second equation to the first and the third one and rearrange:

$$\begin{aligned} \begin{cases} x + 2y + 3z = 1 \\ -x + 3y + 2z = 4 \\ x + 7y + 13z = 6 \end{cases} &\Rightarrow \begin{cases} / + 5y + 5z = 5 \\ -x + 3y + 2z = 4 \\ / + 10y + 15z = 10 \end{cases} \Rightarrow \\ &\Rightarrow \begin{cases} -x + 3y + 2z = 4 \\ 5y + 5z = 5 \\ 10y + 15z = 10 \end{cases} \Rightarrow \begin{cases} -x + 3y + 2z = 4 \\ y + z = 1 \\ 2y + 3z = 2 \end{cases} \end{aligned}$$

Now, we multiply the second equation by 3, and then subtract the third equation from it (then rearrange):

$$\begin{aligned} \begin{cases} -x + 3y + 2z = 4 \\ y + z = 1 \\ 2y + 3z = 2 \end{cases} &\Rightarrow \begin{cases} -x + 3y + 2z = 4 \\ 3y + 3z = 3 \\ 2y + 3z = 2 \end{cases} \Rightarrow \\ &\Rightarrow \begin{cases} -x + 3y + 2z = 4 \\ y + / = 1 \\ 2y + 3z = 2 \end{cases} \Rightarrow \begin{cases} -x + 3y + 2z = 4 \\ 2y + 3z = 2 \\ y = 1 \end{cases} \end{aligned}$$

Now, since it is in the Gauss-Jordan form, we have a unique solution. That solution has $y = 1$.

This solution has the advantage that we don't have to find the other variables x , y . It doesn't use the matrices.

The correct answer is (a) . ■

Second solution :

Form the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & 3 & 2 & 4 \\ 1 & 7 & 13 & 6 \end{array} \right] \xrightarrow{\substack{R_2 + R_1 \\ R_3 - R_1}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 5 & 5 & 5 \\ 0 & 5 & 10 & 5 \end{array} \right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 5 & 5 & 5 \\ 0 & 0 & 5 & 0 \end{array} \right] \Rightarrow z = 0, \text{ and}$$

$$\xrightarrow{\substack{\frac{1}{5} \cdot R_2 \\ \frac{1}{5} \cdot R_3}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right] \Rightarrow y = 1.$$

and we stop here.

The correct answer is (a) . ■

9. Suppose that $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 4 & 2 \end{bmatrix}$.

The matrix $5AA^T$

(a) is $\begin{bmatrix} 50 & 20 \\ 20 & 100 \end{bmatrix}$

(c) cannot be calculated since 5 is not a matrix.

(b) is $\begin{bmatrix} 10 & 4 \\ 4 & 20 \end{bmatrix}$

(d) cannot be calculated since A is not a square matrix.

(e) satisfies none of the above.

Solution :

We have that

$$A^T = \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 0 & 2 \end{bmatrix}.$$

A is a 2×3 matrix and A^T a 3×2 matrix, therefore AA^T can be calculated, and it is a 2×2 matrix. Since AA^T can be calculated, then $5AA^T$ can be calculated, because any matrix can be multiplied by a constant, like the number 5 in this case. Therefore, (c) and (d) are not true.

Now, we compute their product:

$$\begin{aligned} 5AA^T &= 5 \cdot \begin{bmatrix} 3 & 1 & 0 \\ 0 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 0 & 2 \end{bmatrix} = 5 \cdot \begin{bmatrix} 3 \cdot 3 + 1 \cdot 1 + 0 \cdot 0 & 3 \cdot 0 + 1 \cdot 4 + 0 \cdot 2 \\ 0 \cdot 3 + 4 \cdot 1 + 2 \cdot 0 & 0 \cdot 0 + 4 \cdot 4 + 2 \cdot 2 \end{bmatrix} = \\ &= 5 \cdot \begin{bmatrix} 10 & 4 \\ 4 & 20 \end{bmatrix} = \begin{bmatrix} 50 & 20 \\ 20 & 100 \end{bmatrix}. \end{aligned}$$

The correct answer is (a) . ■

10. If possible, find the inverse of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 5 \\ 0 & 1 & 1 \end{bmatrix}$.

The entry in the second row and third column is:

- (a) -1 (c) 5 (e) impossible to determine since the matrix is not invertible.
 (b) 1 (d) -4

Solution :

We use the matrix operations:

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 4 & 5 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_3 - 4 \cdot R_2} \\ & \xrightarrow{R_3 - 4 \cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 \end{array} \right] \xrightarrow{\begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 4 \\ 0 & 1 & 0 & 0 & -1 & 5 \\ 0 & 0 & 1 & 0 & 1 & -4 \end{array} \right] \end{aligned}$$

Therefore, the matrix is invertible, and its inverse is $\begin{bmatrix} 1 & -1 & 4 \\ 0 & -1 & 5 \\ 0 & 1 & -4 \end{bmatrix}$. The entry in the second row and third column is 5 .

The correct answer is (c) . ■

11. Determine k so that there are infinitely many solutions of

$$\begin{cases} 3x - 5y = 12 \\ -x + 2y = k \end{cases}$$

(a) $k = -3$

(c) $k = 12$

(e) cannot be done.

(b) $k = 3$

(d) $k = -12$

Solution :

We have infinitely many solutions when both equations represent the same line. Rearranging the equations, by solving both for y , we have,

$$\begin{cases} y = \frac{3}{5}x - \frac{12}{5} \\ y = \frac{1}{2}x + \frac{1}{2}k \end{cases}$$

They have different slopes. Therefore, they neither represent the same line, nor parallel.

The correct answer is (e) . ■

Second solution :

Construct the system augmented matrix:

$$\left[\begin{array}{cc|c} 3 & -5 & 12 \\ -1 & 2 & k \end{array} \right] \xrightarrow[-R_1]{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & -k \\ 3 & -5 & 12 \end{array} \right] \xrightarrow{R_2 - 3 \cdot R_1} \left[\begin{array}{cc|c} 1 & -2 & -k \\ 0 & 1 & 12 + 3k \end{array} \right]$$

$$\xrightarrow{R_1 + 2 \cdot R_2} \left[\begin{array}{cc|c} 1 & 0 & 24 + 5k \\ 0 & 1 & 12 + 3k \end{array} \right] \Rightarrow \text{we have the unique solution } \begin{cases} x = 24 + 5k \\ y = 12 + 3k \end{cases} .$$

The number k is fixed here, the solution is unique, i.e. x and y are independent as variables (one is not expressed as a function of the other). Therefore, no possible value of k will make the system to have infinitely many solutions.

The correct answer is (e) . ■

12. A simple economy has 2 power industries: oil and gas.

The following input-output matrix describes the power production:

$$A = \begin{array}{cc} & \begin{array}{cc} \text{oil} & \text{gas} \end{array} \\ \begin{array}{c} \text{oil} \\ \text{gas} \end{array} & \begin{bmatrix} 1/10 & 3/10 \\ 2/10 & 1/10 \end{bmatrix} \end{array}$$

Suppose the demand is 150 units of oil and 300 units of gas. What level oil production will satisfy the demand ?

- (a) 150 (c) 260 (e) None of the above
 (b) 300 (d) 420

Solution :

Recall that the identity matrix is denoted by $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Denote by $B = \begin{bmatrix} 150 \\ 300 \end{bmatrix}$ the matrix of the demand. Form the matrix $I - A$:

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/10 & 3/10 \\ 2/10 & 1/10 \end{bmatrix} = \begin{bmatrix} 9/10 & -3/10 \\ -2/10 & 9/10 \end{bmatrix}.$$

Now compute the inverse $(I - A)^{-1}$. First, the determinant of $I - A$ is:

$$\det(I - A) = \frac{9}{10} \cdot \frac{9}{10} - \left(-\frac{3}{10}\right) \cdot \left(-\frac{2}{10}\right) = \frac{81}{100} - \frac{6}{100} = \frac{75}{100} = \frac{3}{4} = 0.75.$$

The inverse $(I - A)^{-1}$ is

$$(I - A)^{-1} = \frac{1}{0.75} \cdot \begin{bmatrix} \frac{9}{10} & -\frac{3}{10} \\ -\frac{2}{10} & \frac{9}{10} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \cdot \frac{9}{10} & \frac{4}{3} \cdot \left(-\frac{3}{10}\right) \\ \frac{4}{3} \cdot \left(-\frac{2}{10}\right) & \frac{4}{3} \cdot \frac{9}{10} \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & -\frac{2}{5} \\ -\frac{4}{15} & \frac{6}{5} \end{bmatrix}$$

The solution of our problem can be find by computing

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = (I - A)^{-1} \cdot B = \begin{bmatrix} \frac{6}{5} & -\frac{2}{5} \\ -\frac{4}{15} & \frac{6}{5} \end{bmatrix} \cdot \begin{bmatrix} 150 \\ 300 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \cdot 150 - \frac{2}{5} \cdot 300 \\ -\frac{4}{15} \cdot 150 + \frac{6}{5} \cdot 300 \end{bmatrix} = \begin{bmatrix} 300 \\ 400 \end{bmatrix}$$

Then $x = 300$ units are needed the for oil production, and $y = 400$ units for the gas production.

The correct answer is (b) . ■

13. The system

$$\begin{cases} x + y + z = 1 \\ 2y + 4z = 6 \end{cases}$$

has solutions given by

$$(a) \begin{cases} x = \text{any number} \\ y = x - 2 \\ z = -2x + 3 \end{cases}$$

$$(b) \begin{cases} x = z - 2 \\ y = 2z + 3 \\ z = \text{any number} \end{cases}$$

$$(c) \begin{cases} x = -2z - 3 \\ y = z - 2 \\ z = \text{any number} \end{cases}$$

$$(d) \begin{cases} x = z - 2 \\ y = -2z + 3 \\ z = \text{any number} \end{cases}$$

(e) None of the above.

Solution :

We have more variables than equations. We will solve for x and y , using z as parameter (it can be any number).

We divide the second equation by 2, and then subtract from the first:

$$\begin{cases} x + y + z = 1 \\ 2y + 4z = 6 \end{cases} \Rightarrow \begin{cases} x + y + z = 1 \\ y + 2z = 3 \end{cases} \Rightarrow \begin{cases} x - z = -2 \\ y + 2z = 3 \end{cases} .$$

Therefore, $x = z - 2$, $y = -2z + 3$, and, as we mentioned above, $z = \text{any number}$.

The correct answer is (d) . ■

Second solution :

We consider the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \end{array} \right] \xrightarrow{\frac{1}{2} \cdot R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{array} \right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} x & y & z & \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{array} \right] .$$

The third column is of the coefficients of the variable z , which will be consider as parameter.

Therefore, $x = z - 2$, $y = -2z + 3$, and, as we mentioned above, $z = \text{any number}$.

The correct answer is (d) . ■

