MATH 210 Exam 2 - Solutions Fall 2004

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1. How many vertices has the region described by

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 4 \\ y &\leq 4 \\ x+y &\geq 5 \end{aligned}$$

(a) 1 (c) 4 (e) 5

(b) 2 (d) 3

Solution :

The diagram of our region is

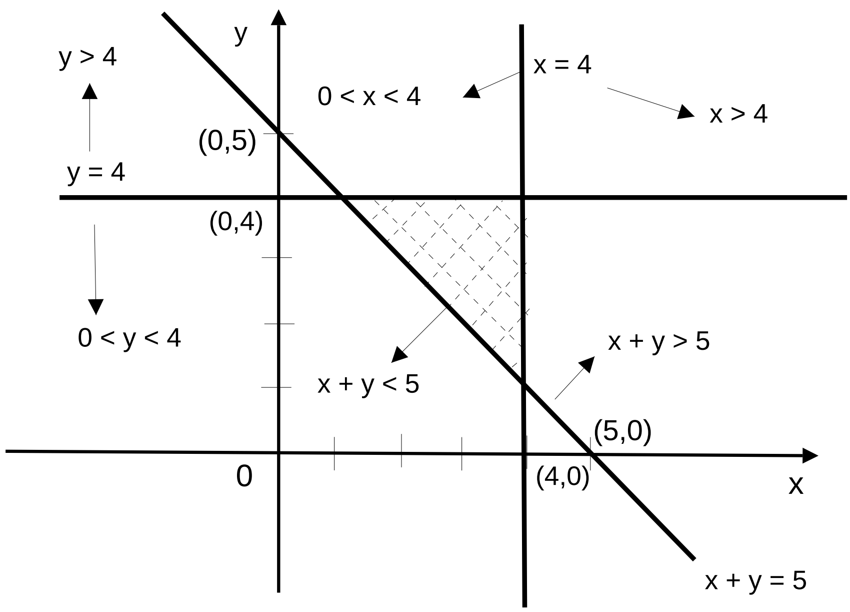

Figure 1: The graphs of $\{x \geq 0 \ , \ y \geq 0\}$, $x \leq 4$, $y \leq 4$ and $x + y \geq 5$.

Our region is a triangle, so it has 3 corners.

The correct answer is (d) . ■

2. Which of the following statements are true ?

I. A primal problem has a solution if and only if the corresponding dual problem has a solution.

II. If a solution exists, then the objective functions of both the primal and the dual problem attain the same value.

III. If a solution exists, then the optimal solution to the dual problem corresponding to the standard maximization problem appears under the slack variables in the last row of the final simplex tableau.

(a) I and II only (b) II only (c) III only

(d) I only (e) all three

Solution :

All three are true.

The correct answer is (e) . ■

3. Which of the following statements are true regarding linear programming problems ?

I. In order to use the Simplex method to minimize a function C, your only choice is to maximize $P = -C$.

II. The Simplex method does not allow any variable to be negative .

III. If the maximum of P occurs at (x, y), then the minimum of $C = -P$ occurs at $(-x, -y)$.

(a) I and II are true (b) I only (c) II only

(d) all three are true (e) None of the above

Solution :

(I) does not describe the correct use of the Simplex method to minimize a function C. First, we construct the dual of the maximization problem, and then C and P are assigned the same value (not $C = -P$!) .

The problem with (III) is that the minimum of $C = -P$ occurs at the same point (x, y), not $(-x, -y)$. It could help you to remember that we require the variables x and y to be positive.

The only true is (II) .

The correct answer is (c) . ■

4. Solve the dual problem

Minimize $C = 2u + 3v$ subject to

$$\begin{aligned} u + 4v &\geq 8 \\ u + v &\geq 5 \\ 2u + v &\geq 7 \\ u, v &\geq 0 \end{aligned}$$

Final tableau

x	y	z	s_1	s_2	P	
0	1	*	*	*	0	5/3
1	0	*	*	*	0	1/3
0	0	2	4	1	1	11

In the **dual** solution

(a) $x = 5/3$

(c) $y = 5/3$

(e) $z = 2$

(b) $v = 4$

(d) $u = 4$

Solution :

The last row tell us that the above table is in the final form: the minimum is reached and it is $C = P = 11$. It is achieved for $s_1 = u = 4$ and $s_2 = v = 1$.

The correct answer is (d) . ■

5. Find the value of the pivot element in the following simplex tableau:

1	0	0	1	0	0	1	0	1
5	10	1	0	0	0	2	0	3
7	2	0	0	1	0	−3	0	2
4	5	0	0	0	1	−6	0	5
−5	−10	0	0	0	0	2	1	4

(a) 1

(b) −3

(c) 4

(d) 2

(e) None of the above

Solution :

The "most" negative number in the last row is −10. Therefore, the second column is the pivot column. Then

1	0	0	1	0	0	1	0	1	→	1/0 not taken
5	10	1	0	0	0	2	0	3	→	3/10
7	2	0	0	1	0	−3	0	2	→	2/2 = 1
4	5	0	0	0	1	−6	0	5	→	5/5 = 1
−5	−10	0	0	0	0	2	1	4		

3/10 is the smallest entry in the last column. Therefore, the pivot is 10 .

The correct answer is (e) . ■

6. Consider the simplex tableau

x	y	z	s_1	s_2	s_3	P	
0	2	0	6	1	3/2	0	44
0	1	1	2/3	0	2	0	8
1	5	0	-2	0	1/3	0	10
0	-8	0	-12	0	4	1	60

The location of the next pivot is :

(a) row 3, column 2

(c) row 1, column 2

(e) none of the above.

(b) row 1, column 4

(d) row 3, column 4

Solution :

The "most" negative entry in the last row is -12. Therefore, the fourth column will be the pivot column. Therefore, the options (a) and (c) are eliminated.

x	y	z	s_1	s_2	s_3	P			
0	2	0	6	1	3/2	0	44	$\longrightarrow$	$44/6 \approx 7.333$
0	1	1	2/3	0	2	0	8	$\longrightarrow$	$8/(2/3) = 24/2 = 12$
1	5	0	-2	0	1/3	0	10	$\longrightarrow$	not applicable, -2 is negative
0	-8	0	-12	0	4	1	60		

Therefore, 6 is the next pivot, and it is situated in the first row and the fourth column.

The correct answer is (b) . ■

7. Consider the simplex tableau

x	y	s_1	s_2	s_3	P	
5	0	12	0	1	0	14
3	0	6	1	0	0	12
8	1	7	0	0	0	5
3	0	4	0	0	1	26

(a) $x = 3, y = 0, P = 0$

(b) $x = 0, y = 5, P = 26$

(c) $x = 14, y = 12, P = 26$

(d) $x = 14, y = 12, P = 0$

(e) None of the above.

Solution :

The tableau is in the final form: the maximum is reached and it is $P = 26$.

The unit columns are the second (this implies $y = 5$), the fourth (this implies $s_2 = 12$) and the fifth (this implies $s_3 = 14$). The other variables are assigned to be zero. Therefore, $x = 0$, $y = 5$ and $P = 26$.

The correct answer is (b) . ■

8. Which statement is **true** regarding linear programming problems ?

(a) It is impossible for the maximum value of an objective function to occur at two different corner points of the feasible set .

(b) Any objective function always has a maximum .

(c) When solving a problem by graphing, the feasible set has at most five corner points.

(d) The minimum value of the objective function, if it exists, will occur at a corner point of the feasible set.

(e) If the feasible set is unbounded, then there is no optimal feasible point.

Solution :

(a) is false, since it is possible for the maximum to occur at two adjoint corner points (hence, the maximum occur at each point of the segment of line between the corner points).

(b) is false, since the set may be unbounded, and we have no maximum in this case.

(c) is false, since the feasible set may have more than five corner points if the number of inequalities is big.

(e) is false, since we have a minimum, but not a maximum.

The correct answer is (d) . ■

9. Which of the following is a corner point of the feasible set for the linear programming problem:

Minimized the objective function $C = 8x + 6y$ subject to the constraints

$$\begin{aligned} 4x + 5y &\leq 200, \\ 6x + 3y &\leq 210, \\ x, y &\geq 0. \end{aligned}$$

(a) $(25, 20)$ (b) $(10, 40)$ (c) $(0, 70)$

(d) $(50, 0)$ (e) None of the above

Solution :

The diagram of our region is on next page.

The points $(10, 40)$ and $(0, 70)$ are not corner points, but $(50, 0)$ is a corner point. But it may be not our choice !

Let us find the intersection point of the lines $4x + 5y = 200$ and $6x + 3y = 210$. For this, solve the system

$$\begin{cases} 4x + 5y = 200 \\ 6x + 3y = 210 \end{cases} .$$

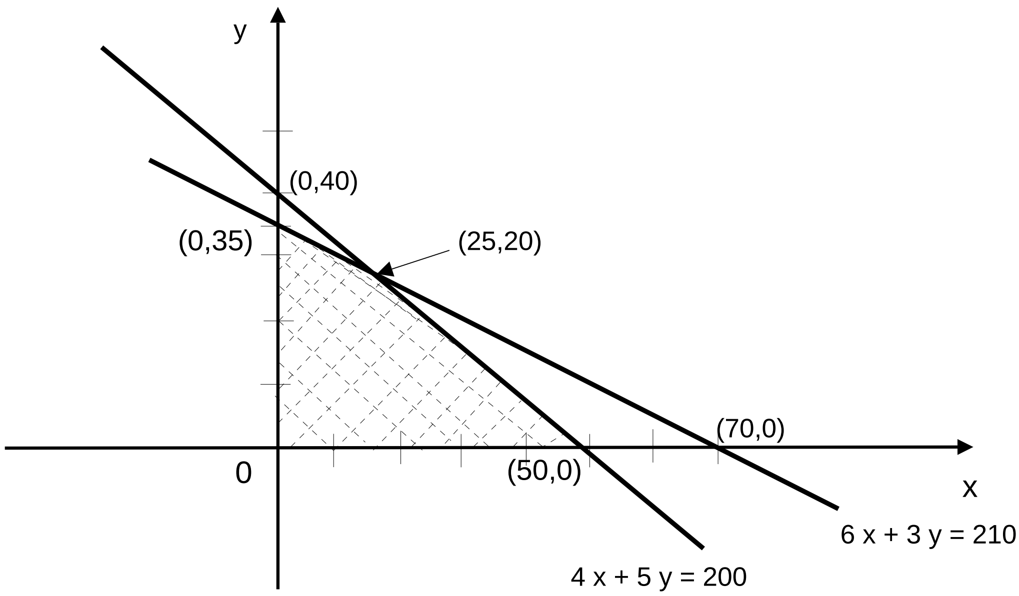

Figure 2: The graphs of $\{x \geq 0\ ,\ y \geq 0\}$, $4x + 5y \leq 200$, and $6x + 3y \leq 210$.

Then, simplify the second equation by dividing it by 3, then multiply it by 2, and,finally, subtract it from the first:

$$\begin{cases} 4x + 5y = 200 \\ 6x + 3y = 210 \end{cases} \Rightarrow \begin{cases} 4x + 5y = 200 \\ 2x + y = 70 \end{cases} \Rightarrow \begin{cases} 4x + 5y = 200 \\ 4x + 2y = 140 \end{cases} \Rightarrow \begin{cases} / + 3y = 60 \\ 4x + 2y = 140 \end{cases} .$$

Therefore, $y = 20$. By substituting it in any equation, let say the first,

$$4x + 5y = 200 \text{ and } y = 20 \Rightarrow 4x + 5 \cdot 20 = 200 \Rightarrow x = 25\ .$$

So $(25, 20)$ is the second corner point we have. The value of $C = 8x + 6y$ at the point $(25, 20)$ is 320, and the value of C at the point $(50, 0)$ is 400 . Since $320 < 400$ and we want to minimize C, we choose $(25, 20)$.

Observation : None of the points $(25, 20)$ and $(50, 0)$ is the solution of our problem, mathematically speaking. The solution is $(0, 0)$, and the minimum C is 0. But $(0, 0)$ is not between our options. Usually, the cost is bigger than 0 if we sell something; in this case, is better to sell 25 for x and 20 for y, than to sell just 50 for x.

The correct answer is (a) . ■

10. Consider the problem

Maximize $P = 7x - 5y + 11z$ subject to

$$\begin{cases} -3x + 5y + 4z & \leq & 25 \\ 4x + 8y + 8z & \leq & 30 \\ x + 2y + 7z & \leq & 10 \\ 2x - y + 9z & \leq & 40 \\ x, y, z & \geq & 0 \end{cases}$$

How many slack variables does the Simplex method use when applied to this problem ?

(a) 2 (b) 5 (c) 4 (d) 1 (e) 3

Solution :

We have four inequality constraints, excepting the constraints $x, y, z \geq 0$. Therefore, we have four slack variables to be used by the Simplex method, like

$$\begin{cases} -3x + 5y + 4z + u & = & 25 \\ 4x + 8y + 8z + v & = & 30 \\ x + 2y + 7z + w & = & 10 \\ 2x - y + 9z + t & = & 40 \\ \text{with } u, v, w, t & \geq & 0 \end{cases}$$

The correct answer is (c) . ■

11. The function, $P = 7x + 8y$, is to be minimized over a triangular shaped region, whose corner points are $A = (2, 9)$, $B = (5, 11)$, and $C = (13, 1)$. What is the minimum value of P ?

(a) 0 (c) 86 (e) None.

(b) 56 (d) 15

Solution :

We know already the corner points of the feasible area. Therefore, we need just to evaluate our function:

(x, y)	P
$(2, 9)$	86
$(5, 11)$	123
$(13, 1)$	99

Therefore, the minimum occurs at $(2, 9)$ and its value is $P_{min} = 86$.

The correct answer is (c) . ■

12. Pivot on the underlined entry 3 in row 1, column 2, in the following tableau:

x	y	s_1	s_2	P	
2	$\underline{3}$	1	0	0	12
1	1	0	1	0	10
−10	−20	0	0	1	0

Which of the following statements are true of the new tableau?

I. Another pivoting is necessary.

II. The optimal value is 80.

(a) II only

(b) I only

(c) neither is true

(d) both are true

(e) None of the above

<u>*Solution*</u> :

Since the pivot is 3 and not 1, we divide first row by 3, and then

x	y	s_1	s_2	P	
2	$\underline{3}$	1	0	0	12
1	1	0	1	0	10
−10	−20	0	0	1	0

$\xrightarrow{\frac{1}{3} R_1}$

x	y	s_1	s_2	P	
$\frac{2}{3}$	1	$\frac{1}{3}$	0	0	4
1	1	0	1	0	10
−10	−20	0	0	1	0

$\xrightarrow[R_3 + 20 \cdot R_1]{R_2 - R_1}$

x	y	s_1	s_2	P	
$\frac{2}{3}$	1	$\frac{1}{3}$	0	0	4
$\frac{1}{3}$	0	$-\frac{1}{3}$	1	0	6
$\frac{10}{3}$	0	$\frac{20}{3}$	0	1	80

All entries in the last row are now positive, so the optimal value is reached and it is $P = 80$. No further pivoting is necessary.

The correct answer is (a) . ■

13. Which of the following is **not** an example of a linear objective function ?

(a) $P = 3x + 5x^2 + 7x^3$ (b) $C = 2x_1 + 3x_2 + 5x_3$ (c) $P = x - 3y + 5z$

(d) $C = -2u + 3v - 5w$ (e) $P = 3x + 5y + 7z$

Solution :

The function $P = 3x+5x^2+7x^3$ from (a) is a cubic function, not a linear function, because its degree is 3. The other are all linear.

The correct answer is (a) . ■

14. A company produces two models of fax machines: a Value and a Deluxe. Each Value model costs \$ 200 to make while each Deluxe costs \$ 300 to make. The profits are \$ 25 for each Value model and \$ 40 for each Deluxe. The total number of fax machines demanded per month does not exceed 2,500 and the company has a budget of \$ 600,000 for manufacturing costs. How many units of each model should the company produce in order to maximize its profits? Suppose that x Value models and y Deluxe models are produced per month.

Which of the following is one of the constraints?

(a) $25x + 40y \leq 600,000$ (c) $25x + 40y \leq 2,500$ (e) None of the above.

(b) $200x + 300y \leq 2,500$ (d) $x + y \leq 2,500$

Solution :

The problem is:

$$\begin{array}{ll} \text{Maximize} & P = 25x + 40y \\ \text{subject to} & 200x + 300y \leq 600,000 \\ & x + y \leq 2,500 \\ & x \geq 0, y \geq 0 \end{array}$$

The objective function to be maximized is the profit: the profits are \$ 25 for each Value model and \$ 40 for each Deluxe. Therefore, it is $P = 25x + 40y$.

The first constraint comes from the available the budget, \$ 600,000 for manufacturing costs. Each Value model costs \$ 200 to make while each Deluxe costs \$ 300 to make. Therefore, $200x + 300y \leq 600,000$.

The second constraint comes from the demand: the total number of fax machines demanded per month does not exceed 2,500. Therefore, $x + y \leq 2,500$.

The correct answer is (d) . ■

15. Which of the following linear programming problems are in standard form as maximization problems?

I. Maximize $P = 7x - 5y$ subject to $\begin{cases} -3x + 5y & \leq & -15 \\ 4x + 8y & \leq & 30 \\ x, y & \geq & 0 \end{cases}$

II. Maximize $P = 2x + 5y$ subject to $\begin{cases} x + 4y & \leq & 20 \\ 6x - 3y & \leq & 60 \\ x, y & \geq & 0 \end{cases}$

III. Maximize $P = 7x - 5y$ subject to $\begin{cases} -3x + 5y & \geq & 15 \\ 4x + 8y & \leq & 30 \\ x, y & \geq & 0 \end{cases}$

(a) II only

(b) I only

(c) III only

(d) II and III only

(e) Some other selection.

<u>*Solution*</u> :

The first assessment, (I), is not in the standard form since the first constraint is less than a negative number.

The third, (III), is also not in the standard form since the first constraint is greater than a positive number. All constraints have to be less or equal than a positive number. If we multiply the first constraint by -1, it becomes less than a number, but a negative one:

$$-3x + 5y \geq 15 \ / \cdot (-1) \Rightarrow 3x - 5y \leq -15 \ .$$

The correct answer is (a) . ■