Observation : This Exam was given in the Fall 2001.

1. Find the point where the lines $4x + 3y = 6$ and $2x - y = 8$ intersect.

   (a) $\left(\frac{3}{2}, 0\right)$ (b) $\left(\frac{3}{4}, 1\right)$ (c) $(2, -4)$ (d) $(4, 0)$ (e) $(3, -2)$

   Solution :

   Solve
   $$\begin{cases} 4x + 3y = 6 \\ 2x - y = 8 \end{cases} ,$$
   by solving the second equation for y and substituting in the first equation:
   $$y = 2x - 8 \Rightarrow 4x + 3 \cdot (2x - 8) = 6 \Rightarrow 4x + 6x - 24 = 6 \Rightarrow 10x = 30 \Rightarrow$$
   $$\Rightarrow x = 3 \text{ , and then } y = 2x - 8 = 2 \cdot 3 - 8 = -2 \Rightarrow (x, y) = (3, -2) \text{ .}$$
   Therefore, the correct answer is e . $\diamond$

2. The slope of the line $5x + 3y = 9$ is :

   (a) $-\frac{5}{3}$ (b) 5 (c) -5 (d) $\frac{5}{3}$ (e) $\frac{3}{5}$

   Solution :

   Solve $5x + 3y = 9$ for y :
   $$5x + 3y = 9 \Rightarrow 3y = 9 - 5x \Rightarrow y = \frac{9}{3} - \frac{5}{3}\,x \Rightarrow y = -\frac{5}{3}\,x + 3 \Rightarrow m = -\frac{5}{3} \text{ .}$$
   Therefore, the correct answer is a . $\diamond$

3. An adult ticket to a sporting event is \$ 15 and a children ticket is \$ 5. If $15,000$ tickets are sold and \$ $195,000$ is obtained, how many adult tickets were sold?

   (a) $3,000$ (b) $5,000$ (c) $12,000$ (d) $10,000$ (e) $13,000$

   Solution :

   Let us denote by x the number of adult tickets and by y the number of children tickets.

   We sold $15,000$ tickets, so $x + y = 15,000$, and we obtained \$ $195,000$, so $15x + 5y = 195,000$. Therefore, we have to solve the system:
   $$\begin{cases} x + y = 15000 \\ 15x + 5y = 195,000 \end{cases}$$

Solve the first equation for y and substitute in the second equation:

$$y = 15,000 - x \Rightarrow 15x + 5 \cdot (15,000 - x) \Rightarrow 15x + 75,000 - 5x = 195,000 \Rightarrow$$

$$\Rightarrow 10x = 195,000 - 75,000 \Rightarrow 10x = 120,000 \Rightarrow x = 12,000 \text{ adult tickets.}$$

We don't need to know what is the value of y, because the question was "how many adult tickets were sold?".

Therefore, the correct answer is c . $\diamond$

4. Find the equation of the line passing through the points $(4, 0)$ and $(8, 5)$.

   (a) $5x + 4y = 20$ (b) $4x + 5y = 16$ (c) $5x - 4y = 20$

   (d) $4x - 5y = 7$ (e) $x + y = 13$

   Solution :

   The slope-intercept form of a line is

   $$y = m\ x + b \text{ , where } m \text{ is the slope, and } b \text{ is the } y\text{-intercept .}$$

   We don't have the y-intercept ! $(4, 0)$ is the x-intercept.

   The slope is:

   $$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{0 - 5}{4 - 8} = \frac{-5}{-4} = \frac{5}{4} \ .$$

   The line becomes $y = \frac{5}{4}\ x + b$. To find b, we take one point and plug in the equation:

   $$\text{take } (4, 0) \Rightarrow 0 = \frac{5}{\cancel{4}} \cdot \cancel{4} + b \Rightarrow b = 0 - 5 = -5 \Rightarrow y = \frac{5}{4}\ x - 5 \text{ (multiply by 4)} \Rightarrow$$

   $$\Rightarrow 4y = 5x - 20 \Rightarrow 5x - 4y = 20 \ .$$

   Therefore, the correct answer is c . $\diamond$

5. Which matrix is obtained after pivoting around the circled element:

   $$\left[\begin{array}{ccc|c} 9 & 4 & 1 & 7 \\ 4 & \textcircled{2} & 6 & 8 \\ 5 & -2 & 1 & -3 \end{array}\right]$$

(a) $\left[\begin{array}{ccc|c} 1 & 0 & -11 & -9 \\ 2 & 1 & 3 & 4 \\ 1 & 0 & -5 & -11 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 17 & 0 & 13 & 23 \\ 2 & 1 & 3 & 4 \\ 9 & 0 & 7 & 5 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 17 & 0 & -13 & 23 \\ 2 & 1 & 3 & 4 \\ 9 & 0 & 7 & 5 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} 1 & 0 & -11 & -9 \\ 2 & 1 & 3 & 4 \\ 9 & 0 & 7 & 5 \end{array}\right]$

(e) $\left[\begin{array}{ccc|c} 1 & 0 & -11 & 9 \\ 4 & 2 & 6 & 8 \\ 9 & 0 & 7 & 5 \end{array}\right]$

Solution :

We reduce only around 2:

$$\left[\begin{array}{ccc|c} 9 & 4 & 1 & 7 \\ 4 & 2 & 6 & 8 \\ 5 & -2 & 1 & -3 \end{array}\right] \xrightarrow{\frac{1}{2} R_2} \left[\begin{array}{ccc|c} 9 & 4 & 1 & 7 \\ 2 & 1 & 3 & 4 \\ 5 & -2 & 1 & -3 \end{array}\right] \underset{R_3 + 2R_2}{\overset{R_1 - 4\ R_2}{\longrightarrow}}$$

$$\underset{R_3 + 2R_2}{\overset{R_1 - 4\ R_2}{\longrightarrow}} \left[\begin{array}{ccc|c} 1 & 0 & -11 & -9 \\ 2 & 1 & 3 & 4 \\ 9 & 0 & 7 & 5 \end{array}\right].$$

and we stop here, because we don't use anymore the initial circled element once **its column** has 1 in its position, and 0 in the other entries .

Therefore, the correct answer is d . ◇

6. Which values of x and y make the following matrices equal?

$$\begin{bmatrix} x & 3 & y \\ y+9 & 9 & 2 \end{bmatrix}, \begin{bmatrix} 2x+1 & 3 & -4 \\ 5 & 9 & 2 \end{bmatrix}$$

(a) $x = 2$, $y = -4$

(b) $x = -4$, $y = 5$

(c) $x = 3$, $y = -4$

(d) $x = -1$, $y = -4$

(e) they cannot be equal.

Solution :

Both matrices have the same type, 2 rows and 3 columns, so we can proceed.

$$\begin{bmatrix} x & 3 & y \\ y+9 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 2x+1 & 3 & -4 \\ 5 & 9 & 2 \end{bmatrix} \Rightarrow \begin{cases} x = 2x+1 \\ 3 = 3 \\ y = -4 \\ y+9 = 5 \\ 9 = 9 \\ 2 = 2 \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = -4 \ . \end{cases}$$

Therefore, the correct answer is d . ◇

7. Let $A = \begin{bmatrix} 5 & -2 \\ 6 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 2 \\ 15 & 7 \end{bmatrix}$.

What is the element in the first row, second column of $A \cdot B$?

(a) -4 (b) -8 (c) 87 (d) 61 (e) 4

Solution :

$$A \cdot B = \left[\begin{array}{cc} 5 & -2 \\ \hline 6 & 3 \end{array}\right] \cdot \left[\begin{array}{c|c} 7 & 2 \\ 15 & 7 \end{array}\right]$$

But remember ! You don't have to compute all elements in the product. To obtain the element in the first row, second column of $A \cdot B$, we have to multiply only the first row of A with the second column of B:

$$5 \cdot 2 + (-2) \cdot 7 = 10 - 14 = -4$$

Therefore, the correct answer is a . $\diamond$

8. Which of the following matrices are invertible?

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 7 \\ 1 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}.$$

(a) A only (b) B only (c) A and C only

(d) B and C only (e) all are invertible

Solution :

A 2×2 matrix $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $\underline{det(X) = ad - bc \neq 0}$.

Therefore, $det(A) = 4 \cdot 1 - 2 \cdot 2 = 0$, $det(B) = 3 \cdot 3 - 7 \cdot 1 = 2 \neq 0$ and $det(C) = 2 \cdot 6 - 4 \cdot 3 = 0$.

Therefore, the correct answer is b . $\diamond$

9. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & 0 \\ 3 & 5 & 4 \end{bmatrix}$, what is the element in the second row, third column of A^{-1}?

(a) 3 (b) -5 (c) 7 (d) 4 (e) -6

Solution :

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 & 1 & 0 \\ 3 & 5 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 3\,R_1]{R_2 + 2\,R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & 2 & 1 & 0 \\ 0 & -1 & -5 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + R_2}$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 - 6\,R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 8 & -5 & -6 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

We stope here, because we obtained already the element in the second row, third column of A^{-1} : -6. We don't need to obtain all elements in the inverse, once we know that any operation we suppose to do next does not affect that entry (all row R_2 will be not modified).

Therefore, the correct answer is e . $\diamond$

10. After Gauss-Jordan elimination on a system of equations, the following matrix is obtained.

$$\begin{array}{c} \quad x \quad\; y \quad z \\ \left[\begin{array}{ccc|c} 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Which of the following statements is false?

(a) If $y = 3$, then $x = 11$

(b) x is a free variable

(c) There are infinitely many solutions

(d) z=3

(e) One solution is $x = 3$, $y = -1$ and $z = 3$

Solution :

The last row of the matrix has only zero entries, so we have 2 equations and 3 variables. The answer (d) is true, we have infinitely many solutions by **Theorem I**, page 100 in the book.

Take a look at the second row of the matrix. The coefficient of the z variable is 1, and the other variables x and y have coefficient 0. Therefore $z = 3$, and (d) is true.

The first row gives us that $x - 2y = 5$. Declare $y = t$, where t is a real number, said to be **a parameter** in this case. Therefore, $x = 5 + 2y = 5 + 2t$. The general solution of the system is expressed in the form :

$$\begin{cases} x = 5 + 2t \\ y = t \\ z = 3\ . \end{cases}$$

Because there exist the possibility that the variable x depends on the variable y (meaning that x changes if y changes), x is **NOT** a free variable. Because $z = 3$, no matter what the values of x and y are, we say that z is a free variable. Since x and y depends on each other, x and y are not free. Therefore (b) is false.

To check quickly (a) : suppose $y = 3$. Then $x = 5 + 2y = 5 + 2 \cdot 3 = 11$, so (a) is true.

(e) is also true, because $x = 3$, $y = -1$ and $z = 3$ is one of the infinitely many solutions (when $t = -1$).

Therefore, the correct answer is b . $\diamond$

11. Suppose that A is a 2×3 matrix, B is a 4×2 matrix, C is a 3×4 matrix, and D is a 4×3 matrix. Which of the following is defined?

(a) $2A + D$

(b) $CB + A$

(c) $AD + D$

(d) $BA + 5C^T$

(e) $CD - AB$

Solution :

(a) Multiplying A by a number, 2, doesn't change its type. Therefore, $2A$ is of type 2×3 and D is of type 4×3. They cannot be added, so $2A + D$ is not defined.

(b) C is of type 3×4 and B is of type 4×2. So, they can be multiplied, and CB is of type 3×2. But A is of type 2×3, so $CB + A$ is not defined.

(c) A is of type 2×3 and D is of type 4×3. So, they cannot be multiplied, and AD is not defined. Therefore, $AD + D$ is not defined.

(d) B is of type 4×2 and A is of type 2×3. So, they can be multiplied, and BA is of type 4×3.
C is of type 3×4, so C^T is of type 4×3. Multiplying C^T by a real number, 5, doesn't change its type.
The matrix $BA + 5C^T$ is defined.

(e) A is of type 2×3 and B is of type 4×2. So, they cannot be multiplied, and AB is not defined. Therefore, $CD - AB$ is not defined.

Therefore, the correct answer is d . ◇

12. In a economy involving agriculture (A) and transportation (T), the input-output matrix is

$$\begin{bmatrix} 0.4 & 0.7 \\ 0.4 & 0.2 \end{bmatrix}.$$

If the consumer demand is for \$ 50 million in agriculture and \$ 100 million in transportation, what gross level of agriculture is needed to meet the demand?

(a) \$ 150 million

(b) \$ 275 million

(c) \$ 40 million

(d) \$ 110 million

(e) \$ 550 million

Solution :

The input-output matrix is actually translated in this form :

$$\begin{array}{cc} & \begin{array}{cc} A & T \end{array} \\ \begin{array}{c} A \\ T \end{array} & \left[\begin{array}{cc} 0.4 & 0.7 \\ 0.4 & 0.2 \end{array}\right] \end{array}$$

This input-output matrix will be denoted by A. Now, let

$$X = \left[\begin{array}{c} x \\ y \end{array}\right]$$

denote the gross production matrix, where x denotes the value of the agricultural products, and y the value of transportation. Also, let

$$D = \left[\begin{array}{c} 50 \\ 100 \end{array}\right]$$

denote the consumer demand (in million).

Then $X = (I - A)^{-1}\ D$, where $I = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$ is the identity matrix.

Therefore,

$$I - A = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] - \left[\begin{array}{cc} 0.4 & 0.7 \\ 0.4 & 0.2 \end{array}\right] = \left[\begin{array}{cc} 0.6 & -0.7 \\ -0.4 & 0.8 \end{array}\right].$$

To compute the inverse of $I - A$, let us remind how you obtain the inverse of a 2×2 matrix. Let $Y = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ such a matrix. The determinant of Y is $det(Y) = ad - bc$, and it is supposed to be non-zero, to have an inverse. Then

$$Y^{-1} = \frac{1}{ad - bc} \left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right].$$

For $I - A$, the determinant is $det(I - A) = 0.6 \cdot 0.8 - (-0.7) \cdot (-0.4) = 0.48 - 0.28 = 0.2$. Then

$$(I - A)^{-1} = \frac{1}{0.2} \left[\begin{array}{cc} 0.8 & 0.7 \\ 0.4 & 0.6 \end{array}\right] = \left[\begin{array}{cc} 4 & 3.5 \\ 2 & 3 \end{array}\right].$$

The last step is

$$X = (I - A)^{-1}\ D = \left[\begin{array}{cc} 4 & 3.5 \\ 2 & 3 \end{array}\right] \cdot \left[\begin{array}{c} 50 \\ 100 \end{array}\right] = \left[\begin{array}{c} 4 \cdot 50 + 3.5 \cdot 100 \\ 2 \cdot 50 + 3 \cdot 100 \end{array}\right] = \left[\begin{array}{c} 550 \\ 400 \end{array}\right]$$

so

$$X = \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 550 \\ 400 \end{array}\right].$$

The answer is $x = \$\ 550$ million for agricultural.

Therefore, the correct answer is e . $\diamond$

13. Given that the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -2 & 3 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 5 & 1 \\ -1 & 3 & 1 \end{bmatrix}$ which of the following is true for this system of equations?

$$\begin{cases} 2x + y - z = 7 \\ x + y - z = 4 \\ -x - 2y + 3z = -2 \end{cases}$$

(a) $y = 6$ (b) $y = 4$ (c) $x = 5$ (d) $z = 4$ (e) $x = -2$

Solution :

Because you have already the inverse of A, the solution will be found in this way:

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot B = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 5 & 1 \\ -1 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 7 + (-1) \cdot 4 + 0 \cdot (-2) \\ (-2) \cdot 7 + 5 \cdot 4 + 1 \cdot (-2) \\ (-1) \cdot 7 + 3 \cdot 4 + 1 \cdot (-2) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

Now, since

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} \Rightarrow \begin{cases} x = 3 \\ y = 4 \\ z = 3 \end{cases},$$

the only true affirmation from $(a) - (e)$ is $y = 4$.

Therefore, the correct answer is b . ◇

14. Find the inverse of the matrix $\begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$.

(a) $\begin{bmatrix} -7 & -3 \\ -5 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -3 \\ -5 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 3 \\ 5 & -7 \end{bmatrix}$

(d) $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

(e) This matrix is singular .

Solution :

Let us remind how you obtain the inverse of a 2×2 matrix.

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such a matrix. The determinant of X is $det(X) = ad - bc$, and it is supposed to be non-zero, to have an inverse. Then

$$X^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} .$$

In our problem, $det(X) = 7 \cdot 2 - 5 \cdot 3 = 14 - 15 = -1$. Therefore,

$$X^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & -3 \\ -5 & 7 \end{bmatrix} = -\begin{bmatrix} 2 & -3 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 5 & -7 \end{bmatrix}.$$

Therefore, the correct answer is c . ◇

15. Which of the following is true:

    (a) Any matrices of the same size can be added.
    (b) Every non-zero matrix is invertible.
    (c) Any two square matrices can be multiplied.
    (d) Any two matrices of the same size can be multiplied.
    (e) The 0 matrix is invertible.

    <u>*Solution :*</u>

    (a) Is true.

    (b) It is not true. For example, $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ is a non-zero matrix but it is not invertible, since $det(A) = 4 \cdot 1 - 2 \cdot 2 = 0$.

    (c) It is not true. For example,

    $A = \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 3 & -11 & 2 \end{bmatrix}$ are both square matrices, but they cannot be multiplied, since A is of size 2×2 and B is of size 3×3.

    (d) It is not true. Let

    $$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

    are of the same size, 2×3, but they cannot be multiplied, because the number of columns of A, 3, is not equal to the number of rows of B, which is 2.

    (e) It is not true. The zero matrix is $\mathbb{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible, since $det(\mathbb{O}) = 0 \cdot 0 - 0 \cdot 0 = 0$.

    Therefore, the correct answer is a . ◇