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Solutions

Ex. 1) Let $f(x) = x + \dfrac{1}{x}$ and $g(x) = \dfrac{x+1}{x+2}$.

(a) Find $f \circ g$.

(b) Find the domain of $f \circ g$.

Solutions :

(a) $f \circ g$ is defined to be $(f \circ g)(x) = f(g(x))$. Therefore, we have

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x+1}{x+2}\right) = \frac{x+1}{x+2} + \frac{1}{\frac{x+1}{x+2}} = \frac{x+1}{x+2} + \frac{x+2}{x+1} =$$

$$= \frac{(x+1)(x+1)}{(x+2)(x+1)} + \frac{(x+2)(x+2)}{(x+1)(x+2)} = \frac{(x+1)^2 + (x+2)^2}{(x+1)(x+2)} = \frac{x^2+2x+1+x^2+4x+4}{(x+1)(x+2)} =$$

$$= \frac{2x^2+6x+5}{(x+1)(x+2)} \quad \text{(observe that } 2x^2+6x+5 \text{ is prime)} \quad .$$

(b) The domain of $f \circ g$ depends on the domain of $g(x)$, and the domain of $f(g(x))$:

$$\text{Domain of } g = \{x \mid x \neq -2\} \text{ and Domain of } f(g(x)) = \{x \mid x \neq -2, x \neq -1\} \Rightarrow$$

$$\Rightarrow \text{Domain of } f \circ g = \{x \mid x \neq -2, x \neq -1\} \quad . \quad \blacksquare$$

Ex. 2) Express the following expression in the form $H(x) = (f \circ g)(x)$.

$$H(x) = \sin^4\left(\sqrt{x}\right)$$

Emphasize the functions f and g . Check your answer . Find the interval on which the function $H(x)$ is continuous at each point .

Solutions :

You have two choices:

- $f(x) = \sin^4 x$ and $g(x) = \sqrt{x}$. Then, check:

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \sin^4\left(\sqrt{x}\right) = H(x) \quad .$$

$g(x) = \sqrt{x}$ is continuous on $[0, +\infty)$ (where g is defined), and $f(x) = \sin^4 x$ is continuous everywhere (because sin and $\sin^4$ are continuous everywhere). Therefore, $H(x)$ is continuous on $[0, \infty)$.

- $f(x) = x^4$ and $g(x) = \sin(\sqrt{x})$. Then, check:

$$(f \circ g)(x) = f(g(x)) = f(\sin(\sqrt{x})) = \sin^4\left(\sqrt{x}\right) = H(x) \quad .$$

$g(x) = \sin(\sqrt{x})$ is continuous on $[0, \infty)$ because $\sqrt{x}$ is continuous on $[0, \infty)$ and sin is continuous everywhere; $g(x)$ is the composition of these two functions. Also $f(x) = x^4$ is continuous everywhere. Therefore, $H(x)$ is continuous on $[0, \infty)$.

Ex. 3) Circle the appropriate answer . No explanation needed .

True False (a) The function $f(x) = \dfrac{x^2 - 1}{x + 1}$ is the same as the function $g(x) = x - 1$.

True False (b) The limit of the function $f(x) = \dfrac{x^2 - 1}{x + 1}$ is the same as the limit of the function $g(x) = x - 1$ when x approaches -1 .

<u>*Solutions*</u> *:*

(a) The answer is FALSE, because $f(x)$ is defined on $\{x \mid x \neq -1\}$, while $g(x)$ is defined everywhere.

(b) $g(x)$ is continuous everywhere, therefore

$$\lim_{x\to-1} g(x) = \lim_{x\to-1} (x - 1) = -1 - 1 = -2 \quad ,$$

and compare to the limit of $f(x)$ at $x = -1$:

$$\lim_{x\to-1} \frac{x^2 - 1}{x + 1} = \left(\text{case } \frac{0}{0} \text{, simplify} \right) = \lim_{x\to-1} \frac{(x-1)\cancel{(x+1)}}{\cancel{x+1}} = \lim_{x\to-1} (x - 1) = -2 \;.$$

The answer yes TRUE, the limits are equal. ■

Ex. 4) (a) State the (ε, δ)-definition of the limit $\lim_{x \to a} f(x) = L$.

(b) Use the (ε, δ)-definition of the limit to prove the following limit .

$$\lim_{x \to 1} (4x - 5) = -1$$

Solutions :

(a) The (ε, δ)-definition of the limit $\lim_{x \to a} f(x) = L$ is at page 93 in the textbook. (Section 2.4, Definition 2)

(b) Using the (ε, δ)-definition of the limit to prove our limit consists in two steps:

- ▶ *Guessing a value for δ.* We want to find a number δ such that

$$|(4x-5)-(-1)| < \varepsilon \quad \text{whenever} \quad 0 < |x-1| < \delta \quad (L = -1, a = 1, f(x) = 4x-5)$$

But $|(4x - 5) - (-1)| = |(4x - 5) + 1| = |4x - 4| = 4|x - 1|$. Therefore, we want

$$4|x-1| < \varepsilon \text{ whenever } 0 < |x-1| < \delta \Rightarrow |x-1| < \frac{\varepsilon}{4} \text{ whenever } 0 < |x-1| < \delta \ ,$$

which make our guess to be $\delta = \dfrac{\varepsilon}{4}$.

- ▶ *Showing that this δ works.* Given $\varepsilon > 0$, choose $\delta = \dfrac{\varepsilon}{4}$.

If $0 < |x - 1| < \delta = \dfrac{\varepsilon}{4}$, then

$$|(4x - 5) - (-1)| = |(4x - 5) + 1| = |4x - 4| = 4|x - 1| < 4 \cdot \left(\frac{\varepsilon}{4}\right) = \varepsilon \quad .$$

Therefore,

$$|(4x-5)-(-1)| < \varepsilon \quad \text{whenever} \quad 0 < |x-1| < \delta \Rightarrow \lim_{x \to 1} (4x-5) = -1 \quad . \quad \blacksquare$$

Ex. 5) Let $f(x)$ be a function .

(a) State the difference quotient definition of the derivative of $f(x)$.

(b) Explain how one can use $f'(a)$ to find the equation of the tangent line to the curve at a point $(a, f(a))$.

Solutions :

(a) The difference quotient definition of the derivative of $f(x)$ is at page 134 in the textbook. (Section 3.2, Formula 2)

(b) The tangent line to the curve $y = f(x)$ at a point $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a. The equation of the tangent line to the curve $y = f(x)$ at a point $(a, f(a))$ is

$$y - f(a) = f'(a)\ (x - a) \quad .$$

See page 128 in the textbook. (Section 3.1) ■

Ex. 6) Evaluate each of the following limits. You must show how you evaluated the limit (the simplifications and the laws) to get full credit. Do not use the (ε, δ)-definition of the limit.

(a) $\lim\limits_{x\to 5} \dfrac{x^2 - 2x - 15}{x - 5}$

(b) $\lim\limits_{h\to 0} \dfrac{\sqrt{h+4} - 2}{h}$

(c) $\lim\limits_{x\to 3} \dfrac{2x - 3}{x^2 - 3x + 2}$

(d) $\lim\limits_{x\to 0} \dfrac{2}{x^2(x-1)}$

Solutions :

(a) $\lim\limits_{x\to 5} \dfrac{x^2 - 2x - 15}{x - 5} = \left(\text{case } \dfrac{0}{0}\ , \text{ simplify}\right) = \lim\limits_{x\to 5} \dfrac{(x-5)(x+3)}{x-5} = \lim\limits_{x\to 5}\ (x+3) = 8.$

(b) $\lim\limits_{h\to 0} \dfrac{\sqrt{h+4} - 2}{h} = \left(\text{case } \dfrac{0}{0}\ , \text{ simplify}\right) = \lim\limits_{h\to 0} \dfrac{\sqrt{h+4} - 2}{h} \cdot \dfrac{\sqrt{h+4}+2}{\sqrt{h+4}+2} =$

$$= \lim_{h\to 0} \frac{h + \not{4} - \not{4}}{h(\sqrt{h+4}+2)} = \lim_{h\to 0} \frac{\not{h}}{\not{h}\ (\sqrt{h+4}+2)} = \lim_{h\to 0} \frac{1}{\sqrt{h+4}+2} = \frac{1}{\sqrt{0+4}+2} = \frac{1}{4}\ .$$

(c) $\lim\limits_{x\to 3} \dfrac{2x - 3}{x^2 - 3x + 2} = \dfrac{2\cdot 3 - 3}{3^2 - 3\cdot 3 + 2} = \dfrac{3}{2}\ ,$

where we applied the quotient law, the sum law and other laws, because limit exists.

(d) $\lim\limits_{x\to 0} \dfrac{2}{x^2(x-1)} = \left(\text{case } \dfrac{1}{0}\ , \text{ no simplification, analyze it!}\right).$

This limit is either $\pm\infty$ or it doesn't exist. You need to check the left-hand and the right-hand limit.

$$\lim_{x\to 0^-} \frac{2}{x^2(x-1)} = \frac{2}{0^-} = -\infty \quad ,$$

because if you take a negative number very close to 0, the fraction $\dfrac{2}{x^2(x-1)}$ is negative. The same with

$$\lim_{x \to 0^+} \frac{2}{x^2(x-1)} = \frac{2}{0^-} = -\infty \quad ,$$

because if you take a positive number very close to 0, the fraction $\dfrac{2}{x^2(x-1)}$ is still negative. Therefore, the limit exists and it is $-\infty$. ■

Ex. 7) Let $\quad f(x) = \dfrac{x}{x+1}$.

(a) Find the derivative $f'(x)$ by using the difference quotient definition of the derivative of $f(x)$.

(b) Find an equation of the tangent line to the curve $y = f(x)$ at the point $(1, \frac{1}{2})$.

Solutions :

We defined all these concepts in exercise 5. Let use them.

(a) To find the derivative $f'(x)$ by using the difference quotient definition of the derivative of $f(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{x+h}{x+h+1} - \dfrac{x}{x+1}}{h} =$$

$$= \lim_{h \to 0} \frac{\dfrac{(x+h)(x+1) - x(x+h+1)}{(x+h+1)(x+1)}}{h} = \lim_{h \to 0} \frac{(x+h)(x+1) - x(x+h+1)}{h(x+h+1)(x+1)} =$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + \cancel{xh} + \cancel{x} + h - \cancel{x^2} - \cancel{xh} - \cancel{x}}{h(x+h+1)(x+1)} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(x+h+1)(x+1)} =$$

$$= \lim_{h \to 0} \frac{1}{(x+h+1)(x+1)} = \frac{1}{(x+0+1)(x+1)} = \frac{1}{(x+1)^2}$$

(b) The an equation of the tangent line to the curve $y = f(x)$ at the point $(1, \frac{1}{2})$ is

$$y - f(a) = f'(a)\ (x - a) \quad .$$

$$f'(x) = \frac{1}{(x+1)^2} \text{ , and } a = 1 \Rightarrow f'(a) = f'(1) = \frac{1}{(1+1)^2} = \frac{1}{4} \quad ,$$

so, at $(1, \frac{1}{2})$,

$$y - f(a) = f'(a)\ (x - a) \Rightarrow y - \frac{1}{2} = \frac{1}{4}\ (x - 1) \Rightarrow y = \frac{1}{4}\ x + \frac{1}{4} \quad .$$ ■

Ex. 8) Let $f(x) = \sqrt{2x-2} - 2x^2 + 3$. Show that there is a root of $f(x)$ (i.e. a value of x with the property that $f(x) = 0$) on the interval $[1, 3]$.

Solutions :

We use the **Intermediate value theorem**. See the Section 2.5 in the textbook, page 109, **Theorem 10** and **Example 8**.

In our case, we need to show that $N = 0$ is between $f(1)$ and $f(3)$. Because $f(x) = \sqrt{2x-2} - 2x^2 + 3$ is continuous on $[1, \infty)$ where it is defined, this will assure that we will find a number c between 1 and 3 such that $f(c) = N = 0$, which means that c is a root of f on $[1, 3]$.

$$f(1) = \sqrt{2 \cdot 1 - 2} - 2 \cdot 1^2 + 3 = \sqrt{0} - 2 + 3 = 1 > 0 \quad ,$$

$$f(3) = \sqrt{2 \cdot 3 - 2} - 2 \cdot 3^2 + 3 = \sqrt{4} - 18 + 3 = -13 < 0 \quad .$$

Therefore, $N = 0$ is between $f(1)$ (which is positive) and $f(3)$ (which is negative), which guarantee the existence of a number c between 1 and 3 such that $f(c) = 0$. ■

Ex. 9) (3 pts. each) Match the following graphs to the choice that fits the best.

i) continuous everywhere
ii) jump discontinuity
iii) infinite discontinuity
iv) removable discontinuity
v) essential discontinuity (other than infinite discontinuity)
vi) nothing applies

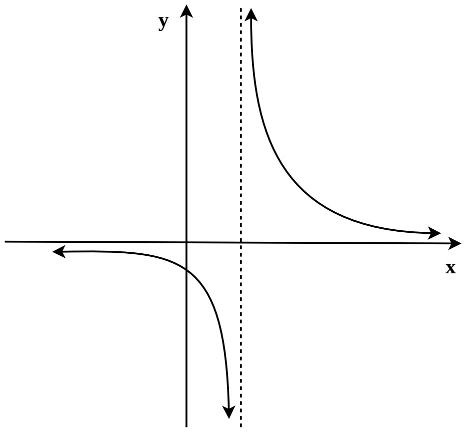
Solutions :

The graph in the first figure (a) show that the functions has a discontinuity at one point, that is an infinite discontinuity (iii).

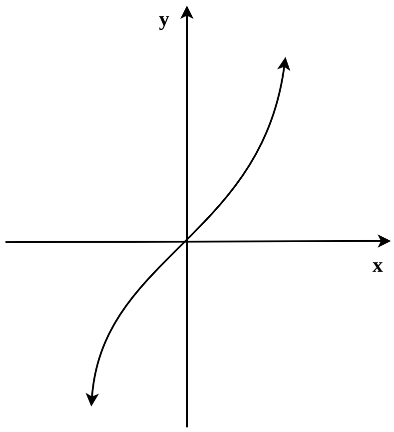
The graph in the first figure (b) show that the functions has no discontinuity at any point, which means that the function is continuous everywhere (i).

The graph in the first figure (c) show that the functions has many discontinuities; this is an example of jump discontinuity (ii).

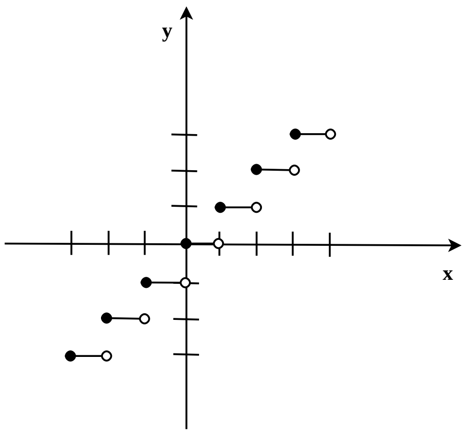
The graph in the first figure (d) show that the functions has a discontinuity at one point, but that point is removable; that is a removable discontinuity (iv).



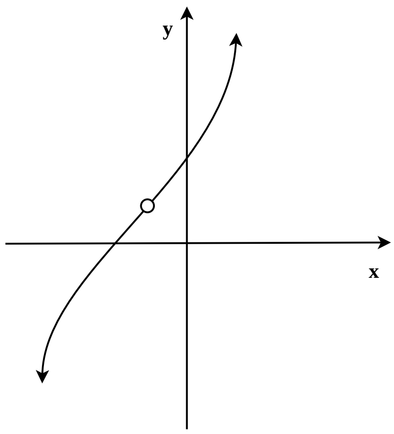
a) ________



b) ________



c) ________



d) ________