# 1. Personal Finance Lecture Notes

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## 2. Percents

- Definition
  - A percent means "so many per hundred."
  - Thirty percent is 30 out of 100.
  - Fifty percent is 50 out of 100, or one half:

$$50\% = \frac{50}{100} = \frac{1}{2}$$

## 3. Warning:

- You need to be careful when you see expressions like 0.04 percent.
- This is not the same as 4 percent, which equals 0.04. You must divide by another hundred:

$$0.04\% = \frac{.04}{100} = .0004$$

- This number is read: four hundredths of one percent.

## 4. Discounts

- A jacket priced at $24 is marked "40% off"
  How much does it cost?
- **Solution:** Multiply $24 by $1 - .4 = .6$:

$$.6 \times 24 = 14.40$$

- Forty percent off means you must pay 60 percent of the original price.

## 5. Big Sale

- A store advertises "Going Out of Business Sale:
  Everything in the Store is "70% off"
- How much does a $50 coat cost?
- **Solution:** Multiply $50 by $1 - .7 = .3$:

$$.3 \times 50 = 15$$

## 6. Additional Savings

- If you see a slip on the coat rack that says

  Take an additional 10% off

  Does this mean that the discount is now 80%?
- **Solution:** No! At 80% off, the coat would cost 20 percent of $50, or $10.
- The store means: (1) take 70 percent off the $50 to get $15; (2) now take 10 percent off the $15.

$$.9 \times 15 = 13.50$$

## 7. A Second Example

- If the store advertises "Take an additional 50% off an item already discounted by 50%,
  you do not get the item for free.
- How much discount are you actually getting?
- **Solution:** The first 50% off, means the item costs .50 times the original price;
  The second 50% off, means the item now costs .50 times the first discounted price:

$$.5 \times .5 = .25$$

- So the effective discount is 75% off the original price.

## 8. Sales Tax

- If the sales tax is 6.5%, what is the final price of a DVD player that is marked \$79.98?
- **Solution:** Multiply 79.98 by $1 + .065 = 1.065$:

$$1.065 \times 79.98 = 85.18$$

- (You usually round up.)
- The restaurant tax in DeKalb is 9%. How much will a \$4.29 extra value meal cost?
- **Solution:** Multiply 4.29 by 1.09: $1.09 \times 4.29 = 4.68$

## 9. Tips

- If you wish to leave a 15% tip on a \$36 meal, how much is the tip?
- **Solution:** Multiply 36 by $.15 = 5.40$:
- **Rule of Thumb:** You can calculate 10% by just moving the decimal point.
  – Ten percent of \$36 is \$3.60.

  - To find twenty percent, just double this:
  - $2 \times 3.60 = 7.20$
  - To find 15 percent, add 10 percent plus half of 10 percent:

$$3.60 + \frac{1}{2}3.60 = 3.60 + 1.80 = 5.40$$

## 10. Eating Out

- If you plan on spending \$40 for two meals at a restaurant, and you need to add on 9% sales tax and a 15% tip, how much money do you need to bring on your date?
- **Solution:** Multiply 40 by both (1.15) and (1.09):

$$(1.15)(1.09)40 = 50.14$$

## 11. Investment and Interest

- You invest \$1000 in the bank at an annual interest rate of 4.5%. How much will you have in your account after one year?
- **Solution:** Multiply 1000 by 1.045: $1.045 \times 1000 = 1045$
- At 4.5% annual interest, how much will you have in your account after 3 years?
- Can you just add 3 times \$45 to the principal?
- **Solution:** No! Because the bank pays you interest on the interest you made in the first and second years.
- This is called compound interest.

## 12. Compound Interest

- At 4.5% annual interest, how much will you have in your account after 3 years?

- Assume the interest is compounded every year.
- You need to multiply 1000 by 1.045 three times, that is, multiply by $(1.045)^3$:
- $(1.045)^3 \times 1000 = 1141.17$
- Since $3 \times 45 = 135$, the interest on the interest added just \$6.17.

## 13. More Compound Interest

- At 4.5% annual interest, how much will you have in your account after 30 years?
- **Solution:** Multiply 1000 by $(1.045)^{30}$:
- $(1.045)^{30} \times 1000 = 3745.32$
- Since $30 \times 45 = 1350$, the principal you would have without compounding the interest would be 2350 instead of 3745.
- You made \$1395 because the interest was compounded, more than the \$1350 in simple interest.

## 14. Monthly Compound Interest

- A bank pays you 6% annual interest, *compunded monthly.* What does this mean?
- **Answer:** They break up the year into 12 monthly periods, and pay you $6/12 = \frac{1}{2}$ percent interest for each period.
- Your account after 1 year is worth

$$(1.005)^{12} \times P = 1.0617 \times P,$$

where $P$ is the beginning principal.

## 15. A.P.R. versus A.P.Y.

- A.P.R. – annual percentage rate

- A.P.Y. – annual percentage yield
- For the previous problem, the A.P.R. is the advertised yearly rate of 6 percent.
- For the previous problem, the A.P.Y. is the actual yearly rate that accrues over the year.
- In this case, we calculated $(1.005)^{12} = 1.0617$
- So the A.P.Y. is 6.17 percent.
- Compounding the interest every month has the same effect as giving an interest rate of 6.17% at the end of the year.

## 16. The National Debt

- As of mid-1998 the national debt was 5.54 trillion dollars.
- In real numbers this is

$$\boxed{5{,}540{,}000{,}000{,}000}$$

- This figure is the money the U.S. government has had to borrow over the years because it has spent more than it has collected in taxes.
- The U.S. government borrows money by selling bonds–mostly treasury bonds, treasury bills, and savings bonds–to anyone who will buy them. (Foreign investors account for 25%.) In return for lending Uncle Sam the money, the bondholders are promised interest on the loan.

## 17. Interest on National Debt

- In 1997 the interest on the national debt was about

$$\boxed{356{,}000{,}000{,}000}$$

- The U.S. government paid more for interest on the national debt than it paid for defense, education, welfare, or any thing else.
- Source: The Arithmetic of Life and Death by George Shaffner

## 18. What each U.S. Citizen Owes

- Assuming there were 270 million U.S. citizens it 1997, how much would each citizen need to contribute in 1997 to pay off the national debt?
- Solution: $\dfrac{5,540,000,000,000}{270,000,000} = 20,518$ dollars per each citizen.
- Assuming there were 142 million tax-paying U.S. citizens in 1997, what was their share of the 356 billion dollar interest on the national debt? How much is this per week?

## 19. National Debt in 2006

Recitation Discussion Problems

- What is the current 2006 national debt? What is the current U.S. population?
- What is each citizen's share of the national debt?
- What is the annual percentage of increase of the national debt over the 8 year period from mid 1998 to now?
- Assuming that the national debt increases at this same rate, because of deficits created by tax cuts and government spending, what will the national debt be in the year 2020?

## 20. Some mildly good news

- The national debt shrinks with population growth, estimated to be about 2 million people per year. Why?
- Its effect is also lessened by inflation. Why?

## 21. Going Both Ways

- A gas station increases prices by 10% because of the increase in fuel prices due to a gas shortage.
- After a while, the shortage is over, and the station decreases prices by 10%.
- Are the prices back to the original prices?
- **Answer:** Surprisingly, no!
  - Suppose the original price is \$1.00.
  - Ten percent of \$1.00 is .10
  - Increase \$1.00 by 10%: \$1.10
  - Ten percent of \$1.10 is .11
  - Decrease \$1.10 by 10%: $1.10 - .11 = .99$
- What happened?

## 22. Mathematical Explanation

- Decreasing by 10% means multiplying by $1 - .1 = .9$
- Increasing by 10% means multiplying by $1 + .1 = 1.1$
- But 1.1 times .9 does not result in 1.
- The correct product is
- $1.1 \times .9 = .99$

## 23. Investment Application

### Aggressive versus Safe

- Dave and Donna both have one thousand dollars to invest over the next three years.
- Dave invests his money in a conservative money market account, which yields a steady 7 percent interest per year.

- Donna invests her money in more volatile "technology stocks," which earn her 30 percent interest in year 1 and year 2, but then lose 30 percent in the third year.
- Who has earned more after 3 years, Dave or Donna?

## 24. Dave's Investment

- **Answer:** Let's start with Dave first. His money has increased by a factor of $1.07^3 = 1.225$
- earning him 22.5 percent interest over 3 years.
- Question: 3 times 7 percent is 21 percent, so where does the extra 1.5 percent interest come from?
- Now consider Donna.
- You might think that the 30 percent loss in year 3 cancels the 30 percent increase in year 2, leaving her roughly the 30 percent increase she made in the first year.
- But it doesn't work like this.

## 25. Donna's Investment

- Her factor of increase is $1.30 \times 1.30 \times 0.70 = 1.183$, earning her 18.3 percent interest, far short of the 30% she made in year one.
- The flaw is that a 30% decrease does not cancel a 30% increase:
- do the math: $1.30 \times 0.70 = .91$,
- resulting in a net loss of 9%.
- Moral: bad years hurt more than good years help.

## 26. Stocks versus Bonds

- Most stock return calculations depend upon an unprecedented bull run between 1982 and 1997, when the stock market went up fourteen of the sixteen years, for an average annual return of almost 15 percent.
- From 1967 to 1982, however, the value of the annual Dow Jones average dropped seven out of fifteen years and the average increased by a total of only five points, from 879 to 884. That equates to an average increase of almost .35 points per year and a average annual return of approximately 0.04 percent.
- Source: The Arithmetic of Life and Death by George Shaffner

## 27. Questions

- What would $2000 invested in mid-1967 be worth in mid-1982?
- What would the same $2000 be worth if invested in CDs yielding an average return of 7%?

## 28. Extrapolating Percentages

- In 1989, Bryant Gumbel, then of the Today Show, interviewed Dr. Charles Hennekens, medical researcher at Harvard, who promoted recent finding confirming the long suspected benefits of aspirin in thwarting heart attacks.
  - The Doctor: Participants in the Harvard Study reduced their risk of heart attacks by 47% by taking one aspirin every other day.
  - Bryant Gumbel: Could you reduce the risk of a heart attack by 94% if you took an aspirin every day?

## 29. Bryant Gumbel Cont'd

- Let's go one step further: If you took one and a half aspirin every day (3 times as much) would you reduce the risk by 141%?
- Note: A 47% reduction means that you now have $.53 = 1 - .47$ times the previous chance of a heart attack. So two such reductions would result in a risk factor of
- $(.53)^2 = .28$
- Can you think of medical considerations?

## 30. Inflation

- Assuming 3 percent inflation over the next 20 years, how much will a 6000 motorcycle cost twenty years from now?
- Solution: $(1.03)^{20} \times 6000 = 10,837$
- At 5 percent inflation, the value increases to $(1.05)^{20} \times 6000 = 15,920$
- If you invest money in a 3 percent certificate of deposit, and the rate of inflation is greater than 3 percent, then you are losing money (in terms of buying power).

## 31. The Value of Education

The following table shows the average income of a person based on her or his level of education:

| Academic Level | 1996 Income | Career Length | Career Earnings |
|---|---|---|---|
| No diploma | 16,536 | 46 years | 1.70 million |
| H.S. diploma | 23,400 | 44 years | 2.37 million |
| Bachelor's Deg | 37,388 | 40 years | 3.68 million |
| Advanced Deg | 47,944 | 37 years | 4.59 million |

## 32. Value of Education

- Source: The Arithmetic of Life and Death by George Shaffner
- The data is adjusted to account for 5 percent inflation, increasing everyone's salary accordingly.
- • Assuming you spend six hours per day going to classes or studying, for 32 weeks a year for four years, how much is your Bachelor's Degree worth to you, based on this chart?
- Solution: Do the math:

$$\frac{3.68 - 2.37 \text{ million dollars}}{4 \text{ years} \times 32 \frac{\text{weeks}}{\text{year}} \times 5 \frac{\text{days}}{\text{week}} \times 6 \frac{\text{hours}}{\text{day}}}$$

$$= 341 \text{ dollars/hour}$$

## 33. Going in Reverse

- Stereo speakers are marked "40% off"
- If the sale price is $100, what is the regular price of the speakers?

- Can you just add 60% of \$100, to compensate for the 40% discount?
  - Since 60% of \$100 is \$60, this method says the original price was \$160.
  - Let's check. What is 40% off \$160?
  - Answer: $.6 \times 160 = 96$, which is not \$100.
  - So the original price was **not** \$160.

## 34. Going in Rerverse

- Stereo speakers are marked "40% off" If the sale price is \$100, what is the regular price of the speakers?
- **Solution:** If $P$ is the original price, then we multiply $P$ by .6 to get the sale price
- $0.6 \times P = 100$
- To find $P$ **we must divide:**
- $P = \frac{100}{.6} = 166.67$

## 35. Sales Tax Example

- Assuming sales tax is 6.5%, what was the original price of a car if the price including the sales tax came to \$13,312.50?
- **Solution:** If $P$ is the original price, then we multiply $P$ by 1.065 to get the final price including tax.
- $1.065 \times P = 13312.50$
- To find $P$ we must divide:
- $P = \frac{13312.50}{1.065} = 12500$

## 36. Interest Example

- You invest a principal at 6% interest.

- If your account is worth \$2650 after one year, how much did you initially invest?
- **Solution:** If $P$ is the original principal, then we multiply $P$ by 1.06 to get the amount in the account after one year.
- $1.06 \times P = 2650$
- To find $P$ we must divide:
- $P = \frac{2650}{1.06} = 2500$

## 37. Compound Interest Example

- You invest a principal at 6% interest, compounded monthly.
- If your account is worth \$3482.30 after one year, how much did you initially invest?
- **Solution:** If $P$ is the original principal, then we multiply $P$ by $(1 + \frac{.06}{12})^{12} = 1.005^{12}$ to get the amount in the account after one year.
- $1.005^{12} \times P = 3482.30$
- To find $P$ we must divide:
- $P = \frac{3482.30}{1.005^{12}} = 3280$

## 38. Savings Bonds

- How much does a \$100 U.S. Savings Bond cost if it matures in five years at 4% interest, compounded annually?.
- **Solution:** If $P$ is the original price of the bond, then we multiply $P$ by $1.04^5$ to get the value of the bond after five years, which is guaranteed to be one hundred dollars:
- $$1.04^5 \times P = 100$$
- To find $P$ we must divide:
- $P = \frac{100}{1.04^5} = 82.19$

- The difference of \$17.81 is the interest you earn for purchasing the bond.

## 39. A Simple Error

- A student from a previous semester once made a simple error in this calculation.
- Instead of writing
- $P = \frac{100}{1.04^5} = 82.19$
- this student wrote
- $P = \frac{100}{0.04^5}$
- omitting the 1 in 1.04
- My calculator says
- $\frac{100}{0.04^5} = 976,562,500$
- Does this answer seem reasonable?

## 40. Saving Money

- Suppose you put $M$ dollars in the bank every month, for six months.
- If the bank pays 0.005 monthly interest (6% annual interest), your money grows like this:

| month | formula | calculated |
|---|---|---|
| 1 | $1.005^5 M$ | $1.0253M$ |
| 2 | $1.005^4 M$ | $1.0202M$ |
| 3 | $1.005^3 M$ | $1.0150M$ |
| 4 | $1.005^2 M$ | $1.0100M$ |
| 5 | $1.005^1 M$ | $1.0050M$ |
| 6 | $M$ | $M$ |
| Total | Sum | $6.0755M$ |

## 41. Final Balance

- After 6 months, you account will be worth

$$1.005^5 M + 1.005^4 M + 1.005^3 M + 1.005^2 M + 1.005^1 M + M$$

- Factoring out an $M$, this sum is
- $(1.005^5 + 1.005^4 + 1.005^3 + 1.005^2 + 1.005^1 + 1)M$
- We want a formula for the sum inside the parentheses.

## 42. Math Saves the Day!

- Let $x = 1.005$ and
- We want to compute
- $S = x^5 + x^4 + x^3 + x^2 + x^1 + 1$
- Recall the formula
- $\boxed{1 + x + x^2 + \cdots x^n = \dfrac{x^{n+1} - 1}{x - 1}}$
- Plugging in $x = 1.005$ and $n = 5$ gives
- $S = \dfrac{x^6 - 1}{x - 1}$
- Note that $x - 1 = 1.005 - 1 = .005$ is just the original monthly interest rate.

## 43. Final Calculation

- For our problem, if you deposit $M = 100$ dollars every month for six months, at a monthly rate of $r = .005$, you would have
- $B = \frac{1.005^6 - 1}{.005} 100 = 607.55$ dollars after 6 months.
- The 7.55 represents accumulated monthly interest.

## 44. Monthly Savings formula

$$\boxed{B = \frac{(1+r)^n - 1}{r} M}$$

where

- $M$ = amount saved per month
- $B$ = ending balance
- $r = i/12$ = monthly interest rate
- $n$ = number of months

## 45. Long Term Savings

- Suppose you saved $M = 100$ dollars every month, for 45 years, at a monthly rate of $r = .005$.
- Here
  - $M = 100$
  - $r = .005$
  - $n = 12 \times 45 = 540$
- After 45 years the account balance would be
- $B = \frac{1.005^{540}-1}{.005} 100 = 275,599$ dollars.
- Your monthly contributions were $540 \times 100 = 54,000$, so most of the growth in the account is due to accrued interest.

## 46. Saving for Retirement

- You and your sister have different ideas about saving for your retirement at age 65.
- At age 25 you start put aside 200 every month into an IRA yielding 7 percent interest.

- After ten years, at age 35, you decide that family obligations require you to save the $200 for college money for your three kids.
- So you stop putting money into the account, but let it continue to collect 7% interest until you retire at age 65.

## 47. Saving for Retirement Cont'd

- Your sister, on the other hand, waits until her 45th birthday to begin saving for her retirement.
- Like you, she has $200 taken out of her paycheck every month into a 7% IRA account.
- At age 65, who has saved more money: you or your sister?

## 48. Your Sister

- Let's look at your sister first, since her situation is easier to analyze.
- The monthly interest rate is $r = .07/12$ for $n = 240$ months.
- Since $M = 200$, the monthly savings formula gives an ending balance after 20 years of
- $\frac{(1+.07/12)^{240}-1}{.07/12}200 = 104,185$
- more than doubling the $240 \times 200 = 48,000$ she has invested.

## 49. Your Turn

- You on the other hand have invested one half as much money as your sister: $60 \times 200 = 24,000$.
- After ten years, when you are 35, your ending balance is computed by the Monthly Savings Formula (with $r = .07/12$ and $n = 120$ months):

- $B = \frac{(1+.07/12)^{120}-1}{.07/12} 200 = 34,616.96$
- Now this money earns compound interest over the next 30 years or $30 \times 12 = 360$ months, so that at age 65 your IRA account will be worth
- $(1 + .07/12)^{360} \times B = 280,968.48$
- You made 2.7 times as much as your sister, although you only invested half as much money.

## 50. Advice

- Moral: How much you accumulate for retirement depends upon three things:
  - (i) when you start saving,
  - (ii) how much you manage to save, and
  - (iii) how much your investments return over the long run.
- Of the three, when you start saving turns out to be the most important.
- Moral: Invest when you are young!

## 51. Monthly Payment $M$ to obtain Balance $B$

$$\boxed{M = \frac{r}{(1+r)^n - 1} B}$$

This formula comes from the Monthly Savings Formula

$$\boxed{B = \frac{(1+r)^n - 1}{r} M}$$

## 52. Saving for College

- Kaylee's parents want to put aside money every month so that their daughter will have $25,000 for college when she turns 18.
- At 5% annual interest, how much do they need to save per month?
- Set the variables:
  - amount saved per month: $M$
  - desired ending balance: $25,000$
  - monthly interest rate: $r = .05/12 = .004166667$
  - number of months: $12 \times 18 = 216$
- $M = \dfrac{.004166667}{(1.004166667)^{216} - 1} 25,000 \quad = 71.60$

## 53. Homework and Quiz

- Homework due next week in Recitation; Conversion sheet exercises, Summation formulas and exercises, Percent exercises
- Miniquiz 2:
  1 pt What's your name, Z-number, and Recitation Section Number?
- 2 pts A standard coke can says
  355 milliliters (ml) = 12 fluid ounces (oz)
  How many milliliters are in 1 fluid ounce?