Math 210 Lecture Notes: Sections 6.3 -6.4 Multiplication principle Permutations Combinations

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1. The Multiplication Principle

To find the number of ways of making several decisions in a row, multiply the numbers of choices that can be made in each decision.

2. True False Tests

How many ways are there to fill out a 2 question True–False Exam?
Since there are 2 ways to answer the first question and 2 ways to answer the second question, there are $2 \times 2 = 4$ possible answer keys:

TT TF FT FF

How many ways are there to fill out a 3 question True–False Exam?
Since there are 2 ways to answer each of the three questions, there are $2^3 = 8$ possible answer keys:

TTT TTF TFT TFF FTT FTF FFT FFF

3. Multiple Choice Tests

How many ways are there to fill out a 2 question Multiple Choice Exam which has 5 possible answers A, B, C, D, E ?
Since there are 5 ways to answer the first question and 5 ways to answer the second question, there are $5 \times 5 = 25$ possible answer keys:

AA	AB	AC	AD	AE
BA	BB	BC	BD	BE
CA	CB	CC	CD	CE
DA	DB	DC	DD	DE
EA	EB	EC	ED	EE

How many ways are there to fill out a 3 question Multiple Choice Exam?
Since there are 5 ways to answer each question, there are $5^3 = 125$ possible answer keys.

4. 20 Question Tests

How many ways are there to fill out a 20 question True False Exam?
Since there are 2 ways to answer each question, there are $2^{20} = 1,048,576$ possible answer keys.
How many ways are there to fill out a 20 question Multiple Choice Exam?
Since there are 5 ways to answer each question, there are $5^{20} = 95,367,431,640,625$ (95 trillion) possible answer keys.

5. License Plates

How many license plates can be made consisting of 3 letters followed by 3 digits?

26	26	26	10	10	10

There are 26 choices for each letter

There are 10 chices for each digit making a total of

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$$

possible license plates.

6. Talking Big

Some people, especially politicians and business types, like to throw big words around. Here is a way to sound important: Select a word from each of the following three lists and combine them.

integrated	management	options
total	organizational	flexibility
systematized	monitored	capability
parallel	reciprocal	mobility
functional	digital	programming
responsive	logistical	concept
optional	transitional	time-phase
synchronized	incremental	projection
compatible	third generation	hardware

7. Talking Big

How many different three–word combinations can be made up in this way?

Since there are 9 words in each list, there are $9^3 = 729$ possible combinations.

8. Genetic Coding

There are four types of small molecules used in genetic coding, represented by their names:

A	adenine
C	cytosine
G	guanine
T	thymine

How many groupings of five molecules are possible?

Since there are 4 choices for each molecule, there are $4^5 = 1024$ possible combinations.

9. Seating Arrangements

There are fourteen people at a formal party. In how many ways can they be seated at the dinner table?

The answer is surprisingly large:

$$87,178,291,200 \quad \text{(87 billion)}$$

How do you get this answer?

Let's start small and come back to this problem.

10. Small Party

There are three people at a small dinner party. Let's call them Andrea, Bill, and Cindy and use the initials A, B, C.

In how many ways can they be seated at the dinner table?

There are 6 ways:

A B C   A C B   B A C   B C A   C A B   C B A

11. Using mult principle

We could use the multiplication priciple:
There are 3 ways to seat the first guest.
Once the first guest is seated, there are two guests still standing, so there are 2 ways to seat the second spot at the dinner table.
Finally the person who is still standing can take the remaining seat in only one way.
By the multiplication principle, there are
$3 \times 2 \times 1 = 6$ ways to seat the three guests.

12. The large dinner party

With 14 guests, the mathematics is still the same.
There are 14 ways to seat the first guest,
13 ways to seat the second chair
12 ways to seat the third chair
etc. down to 1 way to seat the last person standing.
giving a grand total of
$14 \times 13 \times 12 \times \cdots \times 3 \times 2 \times 1$ ways to seat the fourteen guests.
If you are brave enough to multiply these out or have a calculator
the product is 87,178,291,200.

13. Factorial Notation

Numbers such as these occur so often that we give them a name.
$n!$ (pronounced n–factorial) is the product

$$\boxed{n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1}$$

So the number of ways to seat 14 guests at a dinner party is 14!

14. How large is this?

Could a person say all 14! combinations in a lifetime?
The world's fastest speaker speaks at a rate of 595 words per minute.
There are 14 names you need to say for each seating.
Assume each name is a single word and that you speak 16 hours a day, 7 days a week.
Then the time required is

$$87,178,291,200 \times 14 \text{ words} \times \frac{1 \text{ min}}{595 \text{ words}} \times \frac{1 \text{ hour}}{60 \text{ min}} \times \frac{1 \text{ day}}{16 \text{ hours}} \times \frac{1 \text{ year}}{365.25 \text{ days}} = 5850 \text{ years}$$

15. Factorial Growth

Factorials get large quickly as the following table shows:

n	n!
1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880
10	3628800

16. Horse Races

Eight horses are entered in a race. In how many ways can the race result in
(i) win (first)
(ii) place (second) and
(iii) show (third) ?
There are 8 possible horses who can win the race.
Once the winner is established, there are 7 remaining horses who could come in second.
This leave 6 horses competing for third.
We do not care how the horses are ordered after third place.
By the multiplication principle, there are

$$8 \times 7 \times 6 = 336$$

possible outcomes.

17. Permutations

We call the product $8 \times 7 \times 6$ the number of ways of permuting 8 objects taken 3 at a time.
We write this number as $P(8,3)$.
In general, $P(n,r)$ is the number of permutations of n objects taken r at a time.
It has the value

$$P(n,r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)}_{r \text{ factors}}.$$

Note that there are r factors in this product, decreasing from n to $n-r+1$.
In particular $n!$ can be written as the permutation

$$n! = P(n,n)$$

18. Letter Combinations

How many four letter "words" can be made using the standard alphabet?
Without any restrictions, since there are 26 letters in the alphabet, there are 26 choices for each of the four letters. The number of possibilities is:
$26^4 = 456976$
How many four letter "words" can be made using the standard alphabet, where no two letters are the same?
$P(26,4) = 26 \cdot 25 \cdot 24 \cdot 23 = 358800$
How many four letter "words" have at least two letters the same?
Just subtract the answers to the last two questions:
$26^4 - P(26,4) = 456976 - 358800 = 98176$

19. Scrabble

How many four letter "words" can you make using the the letters S C R A M B L E ? (All letters must be distinct.)
$P(8,4) = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$
If you are playing Scrabble and have 7 distinct letters (on your wooden tiles) how many "words" can you form using all these letters?
$P(7,7) = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 5040$
Of course only a few (if any) of these will be actually dictionary words, which is why it is hard to get the 50 bonus points for using all your letters.
What if some of the seven letters are the same?

20. Permutations of Letters

How many ways can you permute the letters: J O E ?
$P(3,3) = 3! = 6$ ways:

$$JOE \quad JEO \quad OJE \quad OEJ \quad EJO \quad EOJ$$

How many ways can you permute the letters: B O B ?
The two B's make this question tricky.
By inspection there are 3 ways:

$$BOB \quad OBB \quad BBO$$

There is another way to determine the number of permutations besides just listing the possibilities.

21. The BOB problem revisited

How many ways can you permute the letters: B O B ?
Pretend for the moment that the B's are different in some way.
Paint one of them green and the other red.
Then there are 6 permutations (just like for JOE):
BOB BBO OBB OBB BBO BOB
But notice that every possible permutation occurs twice in this list:
BOB and BOB
BBO and BBO
OBB and OBB
The correct number is $\frac{6}{2} = 3$

22. Going Bananas

How many ways can you permute the letters: B A N A N A ?
Of the 6 letters, there are 3 A's, 2 N's, and 1 B.
The 2 N's could be rearranged in $2! = 2$ different ways. Like the two B's in BOB.
The 3 A's could be rearranged in $3! = 6$ different ways.
So we need to divide 6! by both 6 and 2.
The number of ways to rearrange the letters in BANANA is
$$\frac{6!}{2!\,3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 2} = 6 \cdot 5 \cdot 2 = 60$$

23. Permutations of Indistinct Objects

The number of ways of permuting n objects in which n_1 objects are of one kind, n_2 objects are of second kind, …, n_k object are of the last kind, where

$$n_1 + n_2 + n_3 + \cdots + n_k = n$$

is

$$\boxed{\frac{n!}{n_1!\, n_2! \cdots n_k!}}$$

24. Ultimate Challenge

How many ways can you permute the letters: M I S S I S S I P P I ?
There are 11 letters of 4 types:

$$n_1 = 4 \text{ I's} \quad n_2 = 4 \text{ S's} \quad n_3 = 2 \text{ P's} \quad n_4 = 1 \text{ M}$$

By the formula, the number of ways to rearrange the letters in MISSISSIPPI is
$$\frac{11!}{4!\,4!\,2!\,1!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{4 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot 2 \cdot 2}$$
$$= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 2} = 11 \cdot 10 \cdot 9 \cdot 7 \cdot 5 = 34650$$

25. Ice Cream Triple Scooper

A true triple-scooper of ice cream is three scoops, each of different flavors, served in a dish.
How many different triples could you get at Baskin-Robbins?
It is well-known that Baskin-Robbins serves 31 different flavors of ice cream.
So it seems reasonable to think that the number of triples is

$$P(31,3) = 31 \times 30 \times 29$$

But wait a minute. It doesn't really matter what order the three scoops are put in the dish.

26. More Ice Cream

For example, if the 3 flavors are chocolate (C), vanilla (V), and strawberry (S) then these could be arranged in $3! = 6$ different ways:
C V S C S V S C V S V C V C S V S C
Any three flavors occur 6 times in the list of $P(31,3)$ permutations.
To get the true number of triple-scoop dishes we must divide by 6:
$\frac{P(31,3)}{3!} = \frac{31 \cdot 30 \cdot 29}{6} = 31 \cdot 5 \cdot 29 = 4495$
This means you could eat a different 3-scoop combination every day for the next 12 years.

27. Combinations

A selection of r out of n objects, without regard to the order the objects are selected is called a combination
In general, $C(n,r)$ is the number of combinations of n objects taken r at a time.
It has the value

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)}{r \cdot (r-1) \cdot (r-2) \cdots 1}$$

Note that both the numerator and denominator contains exactly r factors.

28. Permutations or Combinations

Students are often confused as to whether a problem calls for permutations or combinations.
The rule of thumb is:
use a permutation if the order of selection matters;
use a combination if the order of selection doesn't matter.
Does order matters for the following situations:
Outcome of a race: first, second, or third.
Order matters.
5 card stud poker in which a bet is made after each card is dealt
Order matters.
5 card draw poker in which 5 cards are dealt before the betting begins
Order doesn't matter.

29. Permutations or Combinations II

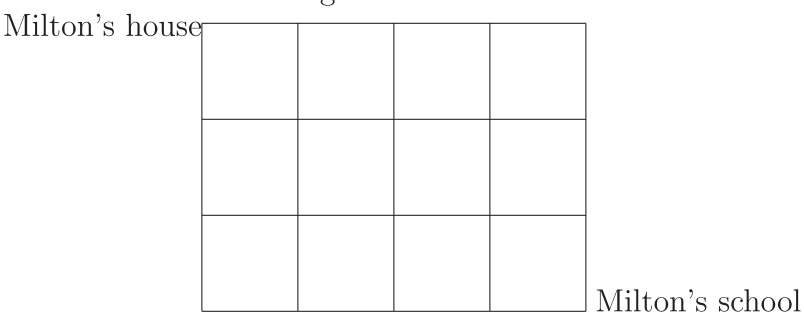
Does order matters for the following situations:
eating 7 courses at a meal
Order matters. As your mother said: you can't eat dessert first.
7 digits selected for a phone number.
Order matters. (753-1835 is different from 753-8531)
3 member of the Math Faculty are selected to form a selection committee for a Finite Mathematics textbook.
Order doesn't matter if each person has equal powers.
Among all NIU Math Majors, 4 officers must be elected to be the President, Vice-President, Treasurer, and Secretary of the Math Club.
Order matters.

30. Poker

How many hands of 5 card stud poker (in which a bet is made after each card is dealt) are there?
$P(52, 5) = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311,875,200$
In how many can 5 cards be dealt in a game of draw poker (in which the betting begins after all 5 cards are dealt)?
$C(52, 5) = \dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 52 \cdot 51 \cdot 10 \cdot 49 \cdot 2$
$= 2,598,960$ or rougly 2 and a half million.

31. Walking to School

Milton lives 4 blocks west and 3 blocks north his school building:



How many different seven block routes can Milton take to school?

32. Milton's Problem Solved

This doesn't seem like an obvious combinatorial problem. But it is!
For each 7–block route, Milton needs to go South out for 3 blocks and East for 4 blocks.
One way of doing this is to put 7 numbers in a hat. Milton draws out 3 of them and elects to go South on these 3 blocks. He walks East on the other 4 blocks.
For example, if Milton drew the numbers 2, 4, 5, then his route to school would be: E S E S S E E.
Since the order in which the 3 southern direction blocks are chosen doesn't matter, the number of 7-block routes to Milton's school is
$C(7, 3) = \dfrac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 7 \cdot 5 = 35$
Milton can walk a different way every month.

33. Prom Picture

Four couples wish to have a prom picture taken of all eight of them. In how many ways can they line up for the picture if each girl stands by her date?
Number the couples C_1, C_2, C_3, C_4.
The number of ways of ordering the four couples is 4!.
Now for each couple, there are two ways they can stand: boy–girl or girl–boy.
Since there are four couples, the solution to the problem is:
$4! \cdot 2^4 = 24 \cdot 16 = 384$

34. The Elevator Problem

Seven people are in an elevator which stops at ten floors. In how many ways can they get off the elevator?
(a) 7^{10}
(b) 10^7
(c) $P(10, 7)$
(d) $C(10, 7)$
(e) None of the above.
There are no restrictions prohibiting two or more people from getting off on the same floor, so the answer is not a permutation of combination. Ruling out (c) and (d), the answer is either (a) or (b). But which one? Do the people choose the floors or do the floors choose the people?

35. The Elevator Problem II

Seven people are in an elevator which stops at ten floors. In how many ways can they get off the elevator?
Each of the seven people $P_1, P_2, \ldots, P_7$ can select which floor to exit:

10	10	10	10	10	10	10
P_1	P_2	P_3	P_4	P_5	P_6	P_7

There are 10 chices for each person P_k

The correct answer is (b) 10^7

36. The Elevator Problem III

Seven people are in an elevator which stops at ten floors. In how many ways can they get off the elevator if no two people get off at the same floor?
(a) 7^{10}
(b) 10^7
(c) $P(10, 7)$
(d) $C(10, 7)$
(e) None of the above.

37. The Elevator Problem IV

Solution:
Number the seven people $P_1, P_2, \ldots, P_7$

10	9	8	7	6	5	4
P_1	P_2	P_3	P_4	P_5	P_6	P_7

There are 10 chices for person P_1
There are 9 remaining chices for person P_2
There are 8 remaining chices for person P_3
There are 7 remaining chices for person P_4
There are 6 remaining chices for person P_5
There are 5 remaining chices for person P_6
There are 4 remaining chices for person P_7
The correct answer is (c) $P(10, 7)$

38. The Elevator Problem V

Seven people are in an elevator which stops at ten floors. No two people get off at the same floor. Before the elevator begins to travel, each of them pushes a button for his or her floor. In how many ways can the elevator buttons be lighted?
(a) 7^{10}
(b) 10^7
(c) $P(10, 7)$
(d) $C(10, 7)$
(e) None of the above.
It doesn't matter what order the buttons were pressed, just which 7 of the 10 floors were selected.
The correct answer is (d) $C(10, 7)$