Math 210 Lecture Notes:
Sections 7.3 -7.5
Conditional Probability
Independence
Bayes Formula

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1. Conditional Probability

Many times we know additional information that affects the calculation of a probability:

- What is the probability that a person voted for Obama if you know he is a registered Republican?
- What is the probability of drawing an ace in poker on your fifth card if you alread have two aces in your hand?
- What is that probability that the a person is guilty of murder if you know the crime was committed by a left-handed person?

  If the person is right handed, the probability is 0.

  If the person is left handed, is the probability is 1?

2. Definition

The probability that B occurs given that A occurs is called the conditional probability of B given A and is written

$$P(B \mid A).$$

3. Example

A fair coin is tossed two times. What is the probability that the second coin is a head if you know that at least one head appears.

The four outcomes are:

H	H	✓
H	T	✓
T	H	✓
T	T	

We have checked those outcomes which satisfy the condition that at least one head occurs. Note that the fourth outcome (T–T) in the table is not checked.

Out of the three events that satisfy this condition (H–H, H–T, T–H) exactly two have heads for the second toss.

P(second toss is a head given at least one head) $= \frac{2}{3}$

4. Lawyers and Liars Cont'd

- There are 100 people at a party.
- Forty are liars.
- Twenty-five are lawyers.
- 15 of the lawyers are liars.

If a person is drawn at random at the party (say to win a door prize), what is the probability that he or she is a lawyer?

Since there are 25 lawyers out of 100 people, the probability is

$$\text{P(lawyer)} = \frac{25}{100} = \frac{1}{4}$$

5. Lawyers and Liars Cont'd

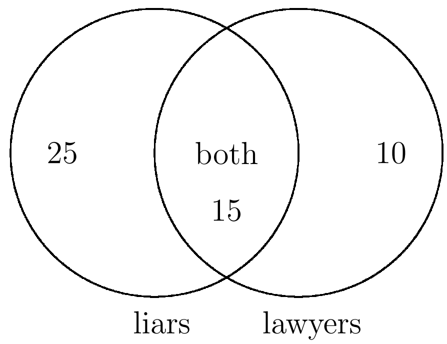
- There are 100 people at a party.
- Forty are liars.
- Twenty-five are lawyers.
- 15 of the lawyers are liars.

If a person is drawn at random at the party, what is the probability that he or she is a lawyer if you know that he or she is a liar?

We are restricted to the subset of forty liars. Since we are told that 15 of these 40 people are also lawyers,

$$P(\text{lawyer} \mid \text{liar}) = \frac{15}{40} = \frac{3}{8}$$

6. Lawyer and Liar Picture



7. A Calculation

By some magic with fractions, $P(\text{lawyer} \mid \text{liar}) = \frac{15}{40}$

$$\begin{aligned} P(\text{lawyer} \mid \text{liar}) &= \frac{15}{40} \frac{\frac{1}{100}}{\frac{1}{100}} \\ &= \frac{\frac{15}{100}}{\frac{40}{100}} \\ &= \frac{P(\text{lawyer and liar})}{P(\text{liar})} \end{aligned}$$

8. Rule for Conditional Probability

The previous example suggests the following rule:

$$\boxed{P(B \mid A) = \frac{P(A \cap B)}{P(A)}}$$

For this equation to make sense, the probability of B must be > 0, that is, we must be certain than event B can happen. Otherwise, we are dividing by zero— and that ain't right!

9. Coin Example

We return to a previous example. A fair coin is tossed two times. What is the probability that the second coin is a head if you know that at least one head appears.

The four outcomes are:

(1) H H (2) H T (3)T H (4) T T

Let A be the event "at least one head" and
B be the even "the second toss is a head"

By direct count

$\mathrm{P}(B) = \frac{2}{4}$

$\mathrm{P}(A) = \frac{3}{4}$

$\mathrm{P}(A \cap B) = \frac{2}{4}$

So by the Conditional Probability Rule

$\mathrm{P}(B \mid A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} = \frac{2/4}{3/4} = \frac{2}{3}$

the same answer we got before.

10. Independence

Two events are called independent if the occurrence or nonoccurrence of one event in no way affects the probability of the second event.

Equivalently, two events A and B are independent if $\mathrm{P}(B \mid A) = \mathrm{P}(B)$

11. Independence Examples

Draw a card from a standard deck.

Let Q be the even the card is a Queen.

Let D be the event the card is a diamond.

Let R be the event the color of the card is red.

$P(Q) = \frac{4}{52}$ There are four queens in the deck.

$P(D) = \frac{13}{52}$ There are 13 diamonds in the deck.

$P(R) = \frac{26}{52}$ There are 26 red cards in the deck.

$P(Q|D) = \frac{1}{13}$ The is exactly one Queen among the 13 diamonds.

$P(D|R) = \frac{13}{26}$ The are 13 diamonds among the 26 read cards.

12. Independence Examples

Example 1. Are Q and D independent?

Since $P(Q|D) = P(Q) = \frac{1}{13}$, Q and R are independent.

If you asked someone whether the card they drew was a Queen, and they replied "I'll give you a hint—it was a diamond," your response should be "Thanks for nothing."

The information that the card is a diamond is useless. Knowing that the card is a diamond does not change the probability of the card being a Queen.

Example 2. Are D and R independent?

Since $P(D|R) = \frac{1}{2} \neq P(D)$, D and R are not independent.

Knowing that the card is red makes it twice as likely that the card is a Diamond.

13. Independence Rule

Using the formula for conditional probability in the definition of independence, we get that

two events A and B are independent if

$$P(B) = P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

or, cross-multiplying,

$P(A \cap B) = P(A) \cdot P(B)$

In words,

> When two events are independent, then the probability that they both occur is the product of their separate probabilities.

14. Flipping a coin two times

The outcome of the second toss is in no way dependent on the outcome of the first toss.

So the probability of landing two heads in a row is the probability of heads on the first toss times the probability of heads on the second toss

$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$

15. Flip a coin twice

Another way to see this is to list the possible outcomes:

first toss	second toss
Heads	Heads
Heads	Tails
Tails	Heads
Tails	Tails

$$\text{Prob(two heads)} = \frac{1}{4}$$

16. Shooting two foul shots

A basketball player shoots two foul shots. Let F be the event that he makes the first foul shot. and S be the event that he makes the second.

Are F and S independent?

There are three thoughts about this problem:

(1) They are independent events. A player's foul shooting average will remain the same for both shots.

(2) A player has a slightly better chance of making the second shot, because if he misses the first, he can "line up" the second more accurately.

(3) A player has a slightly worse chance of making the second shot, because if he misses the first, he will be "rattled" (less confident) shooting the second.

What do you think?

17. The Monty Hall Problem

A contestant on a game show is to select one of three doors. Behind one of the doors is a new car, behind the other two are goats.

The contestant chooses Door Number 1.

Monty Hall, the game show host, who knows the contents of all three doors, opens one of Door 2 and Door 3, say Door 2, and shows the contestant that it contains a goat. Monty Hall then asks the contestant if she wants to change her selection.

Should she switch to Door Number 3?

18. Two Solutions

There are two popular answers to the Monty Hall Problem:

(1) Since the car is behind one of two doors, there is a fifty/fifty chance that it is behind the door I initially selected. So there is no advantage to switch.

(2) There was a two-thirds chance that the car was behind Door 2 or Door 3 when I made my first choice. Revealing the empty door just shows me which of the two doors contains the car, assuming that one of them does. So I increase my chance of winning the car from 1/3 to 2/3 by switching.

Which one is correct?

19. The Monty Hall Solution

Let D_1 be the event that the car is behind Door 1.

Let G be the event that the contestant is shown a Goat.

Assuming the game is fair, $P(D_1) = \frac{1}{3}$

Since the contestant is always shown a goat, no matter whether her original selection was correct or not,

$$P(G \mid D_1) = P(G) = 1$$

It follows that D_1 and G are idependent events. Thus

$$P(D_1 \mid G) = P(D_1) = \tfrac{1}{3}$$

So the probability remains 1/3 that the car is behind Door 1. The probability that the car is behind the unshown door is 2/3. It is better to switch doors.

Solution Number 2 was correct.

20. Non–independent Events

Two events are **not independent** if the probability of one event depends on the occurrence or nonoccurrence of the other event.

Cards. Two cards are drawn from a standard deck without replacing the first card.

The probability that the second card is an ace depends on what the first card was.

- If the first card was also an ace, then 3 aces are left in a deck of 51 remaining cards.
- If the first card was not an ace, then 4 aces are left in a deck of 51 cards.

21. Rule for non independent events

How can we compute $P(A \cap B)$ when A and B are not independent?

Answer: Use the formula for conditional probability: $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

Clearing the denominator,

Product rule:

$$\boxed{P(A \cap B) = P(A) \cdot P(B|A)}$$

In words,

> When two events are not independent, then the probability that they both occur is the product of the probability the first event occurs times the probability the second event occurs assuming that the first has occurred.

22. Cards

Draw two cards from a deck (without replacing the first card).

The probability that both cards are aces is the probability that the first card is an ace times the probability the second card is an ace assuming that the first was an ace

$$= \frac{4}{52} \cdot \frac{3}{51} = \frac{12}{52 \cdot 51} = \frac{4 \cdot 3}{4 \cdot 13 \cdot 3 \cdot 17} = \frac{1}{13 \cdot 17} = \frac{1}{221}$$

23. The Marksmen Problem

Marksman A hits the target 80% of the time.

Marksman B hits the target 70% of the time.

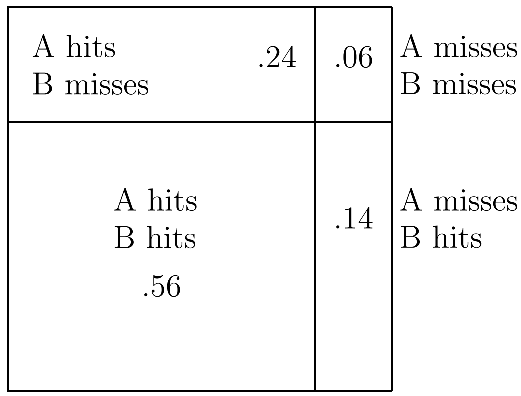
They both fire at the same time.

(1) What is the probability that they both will hit the target?

(2) What is the probability that exactly one of them hits the target?

(3) Given that exactly one bullet hit the target, what is the probability the bullet was fired by A?

24. The Marksmen Problem Picture



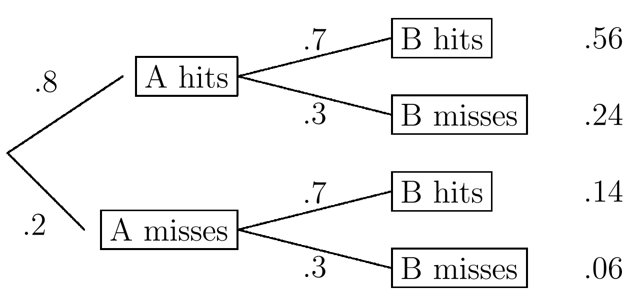
P(A hits and B hits) = P(A hits) P(B hits)
$= (.8)(.7) = 56\%$
P(A hits and B misses) = P(A hits) P(B misses)
$= (.8)(.3) = 24\%$
P(A misses and B hits) = P(A misses) P(B hits)
$= (.2)(.7) = 14\%$
P(A misses and B misses) = P(A misses) P(B misses)
$= (.2)(.3) = 6\%$

P(exactly one hits) = P(A hits, B misses) + P(A misses, B hits)
$= .24 + .14 = .38$

A accounts for 24 of the 38 cases in which exactly one hits the target

So the probability that A hits the target given that exactly one bullet hit the target is $24/38 = 12/19$

25. The Marksmen Diagram



26. Bayes Formula

Two Set Version

A and B are two mutually disjoint sets whose union is the sample space.

$$P(A \mid D)$$
$$= \frac{P(A) \cdot P(D \mid A)}{P(A) \cdot P(D|A) + P(B)P(D|B)}$$

Three Set Version

A, B, and C are 3 mutually disjoint sets (meaning no two of them overlap) whose union is the sample space.

$$P(A \mid D)$$
$$= \frac{P(A) \cdot P(D \mid A)}{P(A)P(D|A)+P(B)P(D|B)+P(C)P(D|C)}$$

27. Marksmen Problem

We coule use Bayes Formula to solve the Marksmen Problem:

- Marksman A hits the target 80% of the time.
- Marksman B hits the target 70% of the time.
- They both fire at the same time.
- Given that exactly one bullet hit the target, what is the probability the bullet was fired by A?
- Let H be the event Marksman A hits the target
- Let M be the event Marksman A misses the target
- Let D be the event exactly one of them hits the target.

28. Marksman Solution ala Bayes

P(H) = P(A hits) = .8

P(M) = P(A misses) = .2

P(D | H) = P(exactly one hits | A hits)

= P(B misses) = .3

P(D | M) = P(exactly one hits| A misses)

= P(B hits) = .7

$$P(\text{H} \mid \text{D}) = \frac{P(\text{H}) \cdot P(\text{D} \mid \text{H})}{P(\text{H})P(\text{D}|\text{H}) + P(\text{M})P(\text{D}|\text{M})}$$
$$= \frac{(.8)(.3)}{(.8)(.3) + (.2)(.7)}$$
$$= \frac{.24}{.24 + .14} = \frac{24}{38}$$

the same answer we got before.

29. Healthcare Solution

Let POS be the event the test is positive.

TB be the event the person actually has TB.

No TB be the event the person does not have TB.

Since the incidence of TB is 5 persons per 10,000,

P(TB) = .0005 and P(No TB) = .9995

Since the chest x-ray detects TB 90 of the time,

P(TB|POS) = .9

Since 1% of the time the test incorrectly indicates the examinee has TB,

P(No TB|POS) = .01

30. Healthcare Solution

P(TB) = .0005 and P(No TB) = .9995

P(TB|POS) = .9 and P(No TB|POS) = .01

$$P(\text{TB} \mid \text{Pos})$$
$$= \frac{P(\text{TB}) \cdot P(\text{POS} \mid \text{TB})}{P(\text{TB})P(\text{POS}|\text{TB}) + P(\text{No TB})P(\text{No TB}|\text{POS})}$$
$$= \frac{(.0005)(0.9)}{(.0005)(0.9) + (.9995)(.01)}$$
$$= \frac{.00045}{.00045 + .009995} = \frac{.00045}{.010445} = .04308$$

Even though the test is 90% accurate, and fails 1% of the time, the probability that a person who tests positive acutally has the disease is a surprisingly low 4.3%

31. Who Dunnit?

Hercule Poirot, a detective,

suspects the butler of committing *MURDER*.

He is 60% sure the butler is guilty.

He then discovers that the crime was committed by a left-handed person and the butler is left-handed.

If only 10% of all people are left–handed, how does this new information change the probabilities?

32. Who Dunnit Solution

Let A be the event the butler is guilty

B be the event the butler is innocent

D be the event the butler is left-handed (LH)

$$P(\text{guilty} \mid \text{LH})$$
$$= \frac{P(\text{guilty}) \cdot P(\text{LH} \mid \text{guilty})}{P(\text{guilty})P(\text{LH}|\text{guilty})+P(\text{innocent})P(\text{LH}|\text{innocent})}$$
$$= \frac{(.60)(1.0)}{(.60)(1.0) + (.40)(.10)}$$
$$= \frac{.60}{.60 + .04}$$
$$= \frac{.60}{.64}$$
$$= .9375$$

33. Search and Rescue

A search and rescue team is looking for a lost hiker in the mountains, last seen at elevation 8,000 feet.

There are three places the hiker could go:
Region 1. higher up the mountain
Region 2. the same elevation near where last seen
Region 3. down the mountain.

There are three types of searches:
(a) thorough search using brigade of interlocked arms
(b) semi-thorough search using dogs
(c) long range helicopter search.

Rates of effectiveness:
(a) 94%
(b) 60%
(c) 25%

34. Search and Rescue

The rescue supervisor assigns probabilities to the three regions:
P(Region 1) = 20%
P(Region 2) = 30%
P(Region 3) = 50%

A thorough search is assigned to Region 3.
Dogs are assigned to search Region 2.
Helicopter is assigned to Region 1.

If the lost hiker is not found, then Bayes Formula gives the probability that he is in each of the three regions.

35. Search and Rescue

The variables needed in Bayes Forula are:

$a = \text{P(Region 1)} \cdot \text{P(not found | Region 1)}$

$b = \text{P(Region 2)} \cdot \text{P(not found | Region 2)}$

$c = \text{P(Region 3)} \cdot \text{P(not found | Region 3)}$

The denominator in Bayes Formula is

$$d = a + b + c$$

By Bayes Formula

$\text{P(Region 1 — not found)} = \dfrac{a}{d}$

$\text{P(Region 2 — not found)} = \dfrac{b}{d}$

$\text{P(Region 3 — not found)} = \dfrac{c}{d}$

All that remains is to plug in the values.

36. Search and Rescue

P(Region 1) = .20 with a 25 % effective search

P(Region 2) = .30 with a 60 % effective search

P(Region 3) = .50 with a 94 % effective search

$a = P(\text{Region 1}) \cdot P(\text{not found | Region 1}) = (.20)(.75) = .15$

$b = P(\text{Region 2}) \cdot P(\text{not found | Region 2}) = (.30)(.40) = .12$

$c = P(\text{Region 3}) \cdot P(\text{not found} \mid \text{Region 3}) = (.50)(.06) = .03$

$d = a + b + c = .15 + .12 + .03 = .30$

P(Region 1 — not found) $= \frac{a}{d} = \frac{.15}{.30} = .50$

P(Region 2 — not found) $= \frac{b}{d} = \frac{.12}{.30} = .40$

P(Region 1 — not found) $= \frac{c}{d} = \frac{.03}{.30} = .10$

37. Summary

Region	Before Search	After Search
1	.20	.50
2	.30	.40
3	.50	.10

For the second round, the Search Supervisor should concentrate
her best team in Region 1
dog team in Region 2 and
use the helicopter in Region 3.

38. A Truel Problem

From Ask Dr. Math, The Math Forum

Here's a puzzler for you all:

You and two of your friends get into a dispute and decide to solve it with a "truel", a three way duel.

Friend #1 is a crack shot, never missing his target. Friend #2 hits his target 2/3 of the time. You hit your target 1/3 of the time. It is decided that you will take the first shot, the 2/3 marksman will take the second shot (if still alive) and the 100% marksman will go last. This will continue until there is only one left alive. On your turn you get to fire one bullet. You get to go first. In order to maximize your chances of living thru this, where should you take your opening shot? And what are your chances of winning the truel if you follow this strategy?

39. A Truel Solution

You should shoot at the ground, i.e., intentionally miss.

Let's say the 100% shot is named MR. White and the 2 out 3 shot is named Mr. Gray.

If you shoot at the ground, it is Mr. Gray's turn. Mr. Gray would rather shoot at Mr. White than you, because he is better. If Mr. Gray kills Mr. White, it is just you and Mr. Gray left, giving you a fair chance of winning. If Mr. Gray does not kill Mr. White, it is Mr. White's turn. He would rather shoot at Mr. Gray and will definitely kill him. Even though it is now you against Mr. White, you has a better chance of winning than before.