1. Percents

- Definition
  - A **percent** means “so many per hundred.”
  - Thirty percent is 30 out of 100.
  - Fifty percent is 50 out of 100, or one half:

\[
50\% = \frac{50}{100} = \frac{1}{2}
\]

2. Warning:

- You need to be careful when you see expressions like 0.04 percent.
- This is not the same as 4 percent, which equals 0.04.
  You must divide by another hundred:

\[
0.04\% = \frac{.04}{100} = .0004
\]
- This number is read: four hundredths of one percent.
3. Discounts

- A jacket priced at $24 is marked “40% off”
  
  How much does it cost?

  **Solution:** Multiply $24 by $1 - .4 = .6:
  
  \[ .6 \times 24 = 14.40 \]

- Forty percent off means you must pay 60 percent of the original price.

5. Additional Savings

- If you see a slip on the coat rack that says
  
  Take an additional 10% off
  
  Does this mean that the discount is now 80%?

  **Solution:** No! At 80% off, the coat would cost 20 percent of $50, or $10.

- The store means: (1) take 70 percent off the $50 to get $15;
  
  (2) now take 10 percent off the $15.

  \[ .9 \times 15 = 13.50 \]

- The percentages multiply:

  Final Price = \(.9 \times .3\) \times\) Original Price

7. Tips

- If you wish to leave a 15% tip on a $36 meal, how much is the tip?

  **Solution:** Multiply 36 by .15 = 5.40:

  **Rule of Thumb:** You can calculate 10% by just moving the decimal point.

  - Ten percent of $36 is $3.60.
  
  - To find twenty percent, just double this:

  \[ 2 \times 3.60 = 7.20 \]

  - To find 15 percent, add 10 percent plus half of 10 percent:

  \[ 3.60 + \frac{1}{2} \times 3.60 = 3.60 + 1.80 = 5.40 \]

8. Eating Out

- If you plan on spending $40 for two meals at a restaurant, and you need to add on 9% sales tax and a 15% tip, how much money do you need to bring on your date?

  **Solution:** Multiply 40 by both (1.15) and (1.09):

  \[(1.15)(1.09)40 = 50.14\]

4. Big Sale

- A store advertises “Going Out of Business Sale:

  Everything in the Store is “70% off”

- How much does a $50 coat cost?

  **Solution:** Multiply $50 by $1 - .7 = .3:

  \[ .3 \times 50 = 15 \]

6. Sales Tax

- If the sales tax is 6.5%, what is the final price of a DVD player that is marked $79.98?

  **Solution:** Multiply 79.98 by $1 + .065 = 1.065:

  \[ 1.065 \times 79.98 = 85.18 \]

- (You usually round up.)

- The restaurant tax in DeKalb is 9%. How much will a $4.29 extra value meal cost?

  **Solution:** Multiply 4.29 by 1.09:

  \[ 1.09 \times 4.29 = 4.68 \]
9. Investment and Interest

- You invest $1000 in the bank at an annual interest rate of 4.5%. How much will you have in your account after one year?
- **Solution:** Multiply 1000 by 1.045:
  \[1.045 \times 1000 = 1045\]
- At 4.5% annual interest, how much will you have in your account after 3 years?
- Can you just add 3 times $45 to the principal?
- **Answer:** It depends on whether the bank is offering simple interest or compound interest.

10. Two Types of Interest

**Simple interest** is computed on the original principal only. At an annual rate of 4.5%, the accrued simple interest on a principal of $1000 after three years will be \[3 \times 45 = 135\].

With **compound interest**, the bank pays you interest on the interest you made in the first and second years.

11. Simple Interest

The formula for simple interest is very simple. Variables:
- \(r\) = annual interest rate
- \(t\) = number of years
- \(P\) = original principal
- \(A\) = amount accumulated after \(t\) years

\[A = P(1 + rt)\]

12. Compound Interest

At 4.5% annual interest, how much will you have in your account after 3 years?
- Assume the interest is compounded every year.
- You need to multiply 1000 by 1.045 three times, that is, multiply by (1.045)^3:
  \[(1.045)^3 \times 1000 = 1141.17\]
- Since \(3 \times 45 = 135\), the interest on the interest added just $6.17.

13. Compound Interest

The formula for compound interest is also simple. Variables:
- \(r\) = annual interest rate
- \(t\) = number of years
- \(P\) = original principal
- \(A\) = amount accumulated after \(t\) years

\[A = P(1 + r)^t\]

14. More Compound Interest

At 4.5% annual interest, how much will you have in your account after 30 years?
- **Solution:** Multiply 1000 by (1.045)^30:
  \[A = (1.045)^{30} \times 1000 = 3745.32\]
- Since \(30 \times 45 = 1350\), the principal you would have without compounding the interest would be 2350 instead of 3745.
- You made $1395 because the interest was compounded, more than the $1350 in simple interest.
15. Moral

- Compound Interest — Good!
- Simple Interest — Bad!

16. Monthly Compound Interest

- A bank pays you 6% annual interest, compounded monthly. What does this mean?
  - Answer: They break up the year into 12 monthly periods, and pay you $6/12 = \frac{1}{2}$ percent interest for each period.
  - Your account after 1 year is worth $(1.005)^{12} \times P = 1.0617 \times P$, where $P$ is the beginning principal.

17. Computing Interest Rates

Your credit card company charges you $4 in one month for a $200 balance. What is your annual interest rate?
  - Answer: Break up the year into 12 months.
  - Your interest rate is $\frac{\text{interest}}{\text{principal}} = \frac{\$4}{\$200} \cdot \frac{1}{\text{month}} \cdot \frac{12 \text{ months}}{1 \text{ year}} = 0.24$ for an annual interest of 24%.

18. Compound Interest Formula

The formula for compound interest with periodic interest conversions per year is:

Variables:
- $r =$ annual interest rate
- $m =$ number of interest periods per year
- $t =$ number of years
- $P =$ original principal
- $A =$ amount accumulated after $t$ years

\[
A = P \left(1 + \frac{r}{m}\right)^{mt}
\]

19. Simplified Formula

The formula $A = P \left(1 + \frac{r}{m}\right)^{mt}$ can be written

\[
A = P (1 + i)^n
\]

where $i = \frac{r}{m}$ and $n = mt$

Here $i$ is the interest rate per period and $n$ is the number of periods.

20. Monthly versus Yearly

- At 4.5% annual interest, the amount accrued in your account after 30 years is yearly $A = (1.045)^{30} \times 1000 = 3745.32$
  - What if the interest was compounded monthly?
  - The monthly interest rate is $i = \frac{.045}{12} = .00375$
  - The number of months is $n = 12 \times 30 = 360$
  - monthly $A = (1.00375)^{360} \times 1000 = 3847.70$
  - It’s worth a little more than $102 to compound monthly versus annually.
21. A.P.R. versus A.P.Y.
• A.P.R. – annual percentage rate
• A.P.Y. – annual percentage yield
• sometimes called the effective rate
• For the previous problem, the A.P.R. is the advertised yearly rate of 6 percent.
• For the previous problem, the A.P.Y. is the actual yearly rate that accrues over the year.
• In this case, we calculated \((1.005)^{12} = 1.0617\)
• So the A.P.Y. is 6.17 percent.
• Compounding the interest every month has the same effect as giving an interest rate of 6.17% at the end of the year.

22. The National Debt
• As of November, 2009 the national debt was 12 trillion dollars.
• In real numbers this is \(12,000,000,000,000\)
• This figure is the money the U.S. government has had to borrow over the years because it has spent more than it has collected in taxes.
• The U.S. government borrows money by selling bonds—mostly treasury bonds, treasury bills, and savings bonds—to anyone who will buy them. (Foreign investors account for 25%.) In return for lending Uncle Sam the money, the bondholders are promised interest on the loan.

23. Interest on National Debt
• In 2008 the interest on the national debt was about \(451,154,000,000\)
• The U.S. government paid more for interest on the national debt than it paid for defense, education, welfare, or any thing else.
• Current Monthly Interest: $36,513,000,000

24. What each U.S. Citizen Owes
• Your Personal Share of the Interest in the National Debt is more than $84,462
• Your share of the annual interest amounts to $261.46 per month.
• Interest figures can fluctuate widely from month to month.
• They are currently very low for the size of the debt, because recent, massive, short-term federal borrowing has been at near zero interest.
• This will change dramatically when the economic recovery begins.
• People and institutions that have parked money in US Treasury instruments will begin to move it into higher yield investments.
25. GOING BOTH WAYS

- A gas station increases prices by 10% because of the increase in fuel prices due to a gas shortage.
- After a while, the shortage is over, and the station decreases prices by 10%.
- Are the prices back to the original prices?
- **Answer:** Surprisingly, no!
  - Suppose the original price is $1.00.
  - Ten percent of $1.00 is .10
  - Increase $1.00 by 10%: $1.10
  - Ten percent of $1.10 is .11
  - Decrease $1.10 by 10%: 1.10 − .11 = .99
- What happened?

26. MATHEMATICAL EXPLANATION

- Decreasing by 10% means multiplying by $1 − .1 = .9$
- Increasing by 10% means multiplying by $1 + .1 = 1.1$
- But 1.1 times .9 does not result in 1.
- The correct product is
- $1.1 \times .9 = .99$

27. INVESTMENT APPLICATION

Aggressive versus Safe

- Dave and Donna both have one thousand dollars to invest over the next three years.
- Dave invests his money in a conservative money market account, which yields a steady 7 percent interest per year.
- Donna invests her money in more volatile “technology stocks,” which earn her 30 percent interest in year 1 and year 2, but then lose 30 percent in the third year.
- Who has earned more after 3 years, Dave or Donna?

28. DAVE’S INVESTMENT

- **Answer:** Let’s start with Dave first. His money has increased by a factor of $1.07^3 = 1.225$
- earning him 22.5 percent interest over 3 years.
  - Question: 3 times 7 percent is 21 percent, so where does the extra 1.5 percent interest come from?
  - Now consider Donna.
  - You might think that the 30 percent loss in year 3 cancels the 30 percent increase in year 2, leaving her roughly the 30 percent increase she made in the first year.
  - But it doesn’t work like this.
29. Donna’s Investment

- Her factor of increase is $1.30 \times 1.30 \times 0.70 = 1.183$, earning her 18.3 percent interest, far short of the 30% she made in year one.
- The flaw is that a 30% decrease does not cancel a 30% increase:
  - do the math: $1.30 \times 0.70 = 0.91$,
  - resulting in a net loss of 9%.
- Moral: bad years hurt more than good years help.

30. Inflation

- Assuming 3 percent inflation over the next 20 years, how much will a 6000 motorcycle cost twenty years from now?
  - Solution: $(1.03)^{20} \times 6000 = 10,837$
- At 5 percent inflation, the value increases to $(1.05)^{20} \times 6000 = 15,920$
- If you invest money in a 3 percent certificate of deposit, and the rate of inflation is greater than 3 percent, then you are losing money (in terms of buying power).

31. Going in Reverse

- Stereo speakers are marked “40% off”
- If the sale price is $100, what is the regular price of the speakers?
- Can you just add 40% of $100, to compensate for the 40% discount?
  - Since 40% of $100 is $40, this method says the original price was $140.
  - Let’s check. What is 40% off $140?
    - Answer: $.6 \times 140 = 84$, which is not $100.
    - So the original price was not $140.

32. Going in Reverse

- Stereo speakers are marked “40% off” If the sale price is $100, what is the regular price of the speakers?
- Solution: If $P$ is the original price, then we multiply $P$ by .6 to get the sale price
  - $0.6 \times P = 100$
  - To find $P$ we must divide: $P = \frac{100}{.6} = 166.67$

33. Sales Tax Example

- Assuming sales tax is 6.5%, what was the original price of a car if the price including the sales tax came to $13,312.50?
- Solution: If $P$ is the original price, then we multiply $P$ by 1.065 to get the final price including tax.
  - $1.065 \times P = 13312.50$
  - To find $P$ we must divide: $P = \frac{13312.50}{1.065} = 12500$

34. Interest Example

- You invest a principal at an effective annual rate of 6% interest.
- If your account is worth $2650 after one year, how much did you initially invest?
  - Solution: If $P$ is the original principal, then we multiply $P$ by 1.06 to get the amount in the account after one year.
    - $1.06 \times P = 2650$
    - To find $P$ we must divide: $P = \frac{2650}{1.06} = 2500$
35. Present versus Future Value

In the formula
\[ A = P(1 + i)^n \]
- \( P \) is called the present value
- \( A \) is called the future value

Sometimes you want to know how much to invest now to obtain an accumulated amount in the future. We can express the present value \( P \) in terms of the future value \( A \) by dividing:
\[ P = \frac{A}{(1 + i)^n} \]

37. Savings Bonds

- How much does a $100 U.S. Savings Bond cost if it matures in five years at 4% interest, compounded annually?
- **Solution:** If \( P \) is the original price of the bond, then we multiply \( P \) by 1.04\(^5\) to get the value of the bond after five years, which is guaranteed to be one hundred dollars:
  \[ 1.04^5 \times P = 100 \]
- To find \( P \) we must divide:
  \[ P = \frac{100}{1.04^5} = 82.19 \]
- The difference of $17.81 is the interest you earn for purchasing the bond.

36. Compound Interest Example

- You invest a principal at 6% interest, compounded monthly.
- If your account is worth $3482.30 after one year, how much did you initially invest?
- **Solution:** If \( P \) is the original principal, then we multiply \( P \) by \((1 + \frac{0.06}{12})^{12} = 1.005^{12}\) to get the amount in the account after one year.
  - \(1.005^{12} \times P = 3482.30\)
  - To find \( P \) we must divide:
    \[ P = \frac{3482.30}{1.005^{12}} = 3280\]

38. A Simple Error

- A student from a previous semester once made a simple error in this calculation.
- Instead of writing
  - \( P = \frac{100}{1.04^5} = 82.19\)
  - this student wrote
    - \( P = \frac{100}{0.04^5} \)
    - omitting the 1 in 1.04
  - My calculator says
    - \( \frac{100}{0.04^5} = 976,562,500\)
  - Does this answer seem reasonable?