1. Percents

• Definition
  – A percent means “so many per hundred.”
  – Thirty percent is 30 out of 100.
  – Fifty percent is 50 out of 100, or one half:

\[
50\% = \frac{50}{100} = \frac{1}{2}
\]

2. Warning:

• You need to be careful when you see expressions like 0.04 percent.
• This is not the same as 4 percent, which equals 0.04.
  
  You must divide by another hundred:

\[
0.04\% = \frac{.04}{100} = .0004
\]

• This number is read: four hundredths of one percent.
3. Discounts

- A jacket priced at $24 is marked “40% off”
  How much does it cost?
- **Solution:** Multiply $24 by \(1 - .4 = .6\):
  \[.6 \times 24 = 14.40\]

- Forty percent off means you must pay 60 percent of the original price.

5. Additional Savings

- If you see a slip on the coat rack that says
  Take an additional 10% off
  Does this mean that the discount is now 80%?
- **Solution:** No! At 80% off, the coat would cost 20 percent of $50, or $10.
  The store means: (1) take 70 percent off the $50 to get $15;
  (2) now take 10 percent off the $15.
  \[.9 \times 15 = 13.50\]

- The percentages multiply:
  Final Price = .9 × .3 × Original Price

7. Tips

- If you wish to leave a 15% tip on a $36 meal, how much is the tip?
- **Solution:** Multiply 36 by .15 = 5.40:
- **Rule of Thumb:** You can calculate 10% by just moving the decimal point.
  - Ten percent of $36 is $3.60.
  - To find twenty percent, just double this:
    \[2 \times 3.60 = 7.20\]
  - To find 15 percent, add 10 percent plus half of 10 percent:
    \[3.60 + \frac{1}{2}3.60 = 3.60 + 1.80 = 5.40\]

4. Big Sale

- A store advertises “Going Out of Business Sale:
  Everything in the Store is “70% off”
- How much does a $50 coat cost?
- **Solution:** Multiply $50 by \(1 - .7 = .3\):
  \[.3 \times 50 = 15\]

6. Sales Tax

- If the sales tax is 6.5%, what is the final price of a DVD player that is marked $79.98?
- **Solution:** Multiply 79.98 by \(1 + .065 = 1.065\):
  \[1.065 \times 79.98 = 85.18\]
- (You usually round up.)
- The restaurant tax in DeKalb is 9%.
  How much will a $4.29 extra value meal cost?
- **Solution:** Multiply 4.29 by 1.09:
  \[1.09 \times 4.29 = 4.68\]

8. Eating Out

- If you plan on spending $40 for two meals at a restaurant, and you need to add on 9% sales tax and a 15% tip, how much money do you need to bring on your date?
- **Solution:** Multiply 40 by both (1.15) and (1.09):
  \[(1.15)(1.09)40 = 50.14\]
9. **Investment and Interest**

- You invest $1000 in the bank at an annual interest rate of 4.5%. How much will you have in your account after one year?
- **Solution:** Multiply 1000 by 1.045: 
  \[ 1.045 \times 1000 = 1045 \]
- At 4.5% annual interest, how much will you have in your account after 3 years?
- Can you just add 3 times $45 to the principal?
- **Answer:** It depends on whether the bank is offering simple interest or compound interest.

10. **Two Types of Interest**

*Simple interest* is computed on the original principal only.
At an annual rate of 4.5%, the accrued simple interest on a principal of $1000 after three years will be \(3 \times 45 = 135\).
With **compound interest**, the bank pays you interest on the interest you made in the first and second years.

11. **Simple Interest**

The formula for simple interest is very simple.
Variables:
- \(r\) = annual interest rate
- \(t\) = number of years
- \(P\) = original principal
- \(A\) = amount accumulated after \(t\) years

\[
A = P(1 + rt)
\]

12. **Compound Interest**

- At 4.5% annual interest, how much will you have in your account after 3 years?
- Assume the interest is compounded every year.
- You need to multiply 1000 by 1.045 three times, that is, multiply by \((1.045)^3\):
  \[
  (1.045)^3 \times 1000 = 1141.17
  \]
- Since \(3 \times 45 = 135\), the interest on the interest added just $6.17.

13. **Compound Interest**

- At 4.5% annual interest, how much will you have in your account after 30 years?
- **Solution:** Multiply 1000 by \((1.045)^{30}\):
  \[
  A = (1.045)^{30} \times 1000 = 3745.32
  \]
- Since \(30 \times 45 = 1350\), the principal you would have without compounding the interest would be 2350 instead of 3745.
- You made $1395 because the interest was compounded, more than the $1350 in simple interest.
15. **Moral**
- Compound Interest — *Good!*
- Simple Interest — *Bad!*

16. **Monthly Compound Interest**
- A bank pays you 6% annual interest, *compounded monthly*. What does this mean?
- **Answer:** They break up the year into 12 monthly periods, and pay you $6/12 = \frac{1}{2}$ percent interest for each period.
- Your account after 1 year is worth $(1.005)^{12} \times P = 1.0617 \times P$, where $P$ is the beginning principal.

17. **Computing Interest Rates**
Your credit card company charges you $4 in one month for a $200 balance. What is your annual interest rate?

**Answer:** Break up the year into 12 months.

Your interest rate is

\[
\frac{\text{interest}}{\text{principal}} = \frac{\$4}{\$200} \times \frac{1 \text{ month}}{1 \text{ year}} = 0.24
\]

for an annual interest of 24%.

18. **Compound Interest Formula**
The formula for compound interest with periodic interest conversions per year is:

Variables:
- $r =$ annual interest rate
- $m =$ number of interest periods per year
- $t =$ number of years
- $P =$ original principal
- $A =$ amount accumulated after $t$ years

\[
A = P \left(1 + \frac{r}{m}\right)^{mt}
\]

19. **Simplified Formula**
The formula $A = P \left(1 + \frac{r}{m}\right)^{mt}$ can be written

\[
A = P(1 + i)^n
\]

where $i = \frac{r}{m}$ and $n = mt$

Here $i$ is the interest rate per period and $n$ is the number of periods.

20. **Monthly versus Yearly**
- At 4.5% annual interest, the amount accrued in your account after 30 years is yearly $A = (1.045)^{30} \times 1000 = 3745.32$
- What if the interest was compounded monthly?
  - The monthly interest rate is $i = \frac{0.045}{12} = .00375$
  - The number of months is $n = 12 \times 30 = 360$
  - monthly $A = (1.00375)^{360} \times 1000 = 3847.70$
- It’s worth a little more than $102 to compound monthly versus annually.
21. A.P.R. versus A.P.Y.
- A.P.R. – annual percentage rate
- A.P.Y. – annual percentage yield
- sometimes called the effective rate
- For the previous problem, the A.P.R. is the advertised yearly rate of 6 percent.
- For the previous problem, the A.P.Y. is the actual yearly rate that accrues over the year.
- In this case, we calculated $1.0617 = (1.005)^{12}$
- So the A.P.Y. is 6.17 percent.
- Compounding the interest every month has the same effect as giving an interest rate of 6.17% at the end of the year.

22. The National Debt
- As of November, 2009 the national debt was 12 trillion dollars.
- In real numbers this is
$$12,000,000,000,000$$
- This figure is the money the U.S. government has had to borrow over the years because it has spent more than it has collected in taxes.
- The U.S. government borrows money by selling bonds—mostly treasury bonds, treasury bills, and savings bonds—to anyone who will buy them. (Foreign investors account for 25%.)
- In return for lending Uncle Sam the money, the bondholders are promised interest on the loan.

23. Interest on National Debt
- In 2008 the interest on the national debt was about
$$451,154,000,000$$
- The U.S. government paid more for interest on the national debt than it paid for defense, education, welfare, or any thing else.
- Current Monthly Interest: $36,513,000,000

24. What each U.S. Citizen Owes
- Your Personal Share of the Interest in the National Debt is more than $84,462
- Your share of the annual interest amounts to $261.46 per month.
- Interest figures can fluctuate widely from month to month.
- They are currently very low for the size of the debt, because recent, massive, short-term federal borrowing has been at near zero interest.
- This will change dramatically when the economic recovery begins.
- People and institutions that have parked money in US Treasury instruments will begin to move it into higher yield investments.
25. Going Both Ways

- A gas station increases prices by 10% because of the increase in fuel prices due to a gas shortage.
- After a while, the shortage is over, and the station decreases prices by 10%.
- Are the prices back to the original prices?
- Answer: Surprisingly, no!
  - Suppose the original price is $1.00.
  - Ten percent of $1.00 is .10
  - Increase $1.00 by 10%: $1.10
  - Ten percent of $1.10 is .11
  - Decrease $1.10 by 10%: $1.10 \times .9 = .99
- What happened?

26. Mathematical Explanation

- Decreasing by 10% means multiplying by $1 - .1 = .9$
- Increasing by 10% means multiplying by $1 + .1 = 1.1$
- But $1.1 \times .9$ does not result in 1.
- The correct product is $1.1 \times .9 = .99$

27. Investment Application

  Aggressive versus Safe

- Dave and Donna both have one thousand dollars to invest over the next three years.
- Dave invests his money in a conservative money market account, which yields a steady 7 percent interest per year.
- Donna invests her money in more volatile “technology stocks,” which earn her 30 percent interest in year 1 and year 2, but then lose 30 percent in the third year.
- Who has earned more after 3 years, Dave or Donna?

28. Dave’s Investment

- Answer: Let’s start with Dave first. His money has increased by a factor of $1.07^3 = 1.225$
- earning him 22.5 percent interest over 3 years.
- Question: 3 times 7 percent is 21 percent, so where does the extra 1.5 percent interest come from?
- Now consider Donna.
- You might think that the 30 percent loss in year 3 cancels the 30 percent increase in year 2, leaving her roughly the 30 percent increase she made in the first year.
- But it doesn’t work like this.
29. Donna’s Investment

- Her factor of increase is $1.30 \times 1.30 \times 0.70 = 1.183$, earning her 18.3 percent interest, far short of the 30% she made in year one.
- The flaw is that a 30% decrease does not cancel a 30% increase:
- do the math: $1.30 \times 0.70 = 0.91$,
- resulting in a net loss of 9%.
- Moral: bad years hurt more than good years help.

30. Inflation

- Assuming 3 percent inflation over the next 20 years, how much will a 6000 motorcycle cost twenty years from now?
- **Solution:** $(1.03)^{20} \times 6000 = 10,837$
- At 5 percent inflation, the value increases to $(1.05)^{20} \times 6000 = 15,920$
- If you invest money in a 3 percent certificate of deposit, and the rate of inflation is greater than 3 percent, then you are losing money (in terms of buying power).

31. Going in Reverse

- Stereo speakers are marked “40% off”
- If the sale price is $100, what is the regular price of the speakers?
- Can you just add 40% of $100, to compensate for the 40% discount?
  - Since 40% of $100 is $40, this method says the original price was $140.
  - Let’s check. What is 40% off $140?
  - Answer: $.6 \times 140 = 84$, which is not $100.
  - So the original price was **not** $140.

32. Going in Reverse

- Stereo speakers are marked “40% off” If the sale price is $100, what is the regular price of the speakers?
- **Solution:** If $P$ is the original price, then we multiply $P$ by $.6$ to get the sale price
  - $.6 \times P = 100$
  - To find $P$ we must divide:
  - $P = \frac{100}{.6} = 166.67$

33. Sales Tax Example

- Assuming sales tax is 6.5%, what was the original price of a car if the price including the sales tax came to $13,312.50?
- **Solution:** If $P$ is the original price, then we multiply $P$ by 1.065 to get the final price including tax.
  - $1.065 \times P = 13312.50$
  - To find $P$ we must divide:
  - $P = \frac{13312.50}{1.065} = 12500$

34. Interest Example

- You invest a principal at an effective annual rate of 6% interest.
- If your account is worth $2650 after one year, how much did you initially invest?
- **Solution:** If $P$ is the original principal, then we multiply $P$ by 1.06 to get the amount in the account after one year.
  - $1.06 \times P = 2650$
  - To find $P$ we must divide:
  - $P = \frac{2650}{1.06} = 2500$
35. Present versus Future Value

In the formula

\[ A = P(1 + i)^n \]

- \( P \) is called the present value
- \( A \) is called the future value

Sometimes you want to know how much to invest now to obtain an accumulated amount in the future.

We can express the present value \( P \) in terms of the future value \( A \) by dividing:

\[ P = \frac{A}{(1 + i)^n} \]

36. Compound Interest Example

- You invest a principal at 6% interest, compounded monthly.
- If your account is worth $3482.30 after one year, how much did you initially invest?

**Solution:** If \( P \) is the original principal, then we multiply \( P \) by \((1 + \frac{0.06}{12})^{12} = 1.005^{12}\) to get the amount in the account after one year.

- \(1.005^{12} \times P = 3482.30\)
- To find \( P \) we must divide:
  - \( P = \frac{3482.30}{1.005^{12}} = 3280\)

37. Savings Bonds

- How much does a $100 U.S. Savings Bond cost if it matures in five years at 4% interest, compounded annually?

**Solution:** If \( P \) is the original price of the bond, then we multiply \( P \) by \(1.04^5\) to get the value of the bond after five years, which is guaranteed to be one hundred dollars:

- \(1.04^5 \times P = 100\)
- To find \( P \) we must divide:
  - \( P = \frac{100}{1.04^5} = 82.19\)
- The difference of $17.81 is the interest you earn for purchasing the bond.

38. A Simple Error

- A student from a previous semester once made a simple error in this calculation.
- Instead of writing
  - \( P = \frac{100}{1.04^5} = 82.19\)
- this student wrote
  - \( P = \frac{100}{0.04^5}\)
- omitting the 1 in 1.04
- My calculator says
  - \( \frac{100}{0.04^5} = 976,562,500\)
- Does this answer seem reasonable?