Math 210 Lecture Notes: Chapter 5.1 Percents Tax Savings Compound Interest

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1. Percents

- Definition
  - A percent means "so many per hundred."
  - Thirty percent is 30 out of 100.
  - Fifty percent is 50 out of 100, or one half:

$$50\% = \frac{50}{100} = \frac{1}{2}$$

2. Warning:

- You need to be careful when you see expressions like 0.04 percent.
- This is not the same as 4 percent, which equals 0.04.
  You must divide by another hundred:

$$0.04\% = \frac{.04}{100} = .0004$$

- This number is read: four hundredths of one percent.

3. Discounts

- A jacket priced at \$24 is marked "40% off"
  How much does it cost?
- **Solution:** Multiply \$24 by $1 - .4 = .6$:

$$.6 \times 24 = 14.40$$

- Forty percent off means you must pay 60 percent of the original price.

4. Big Sale

- A store advertises "Going Out of Business Sale:
  Everything in the Store is "70% off"
- How much does a \$50 coat cost?
- **Solution:** Multiply \$50 by $1 - .7 = .3$:

$$.3 \times 50 = 15$$

5. Additional Savings

- If you see a slip on the coat rack that says

  Take an additional 10% off

  Does this mean that the discount is now 80%?
- **Solution:** No! At 80% off, the coat would cost 20 percent of \$50, or \$10.
- The store means: (1) take 70 percent off the \$50 to get \$15;
  (2) now take 10 percent off the \$15.

$$.9 \times 15 = 13.50$$

- The percentages multiply:
  Final Price $= .9 \times .3 \times$ Original Price

6. Sales Tax

- If the sales tax is 6.5%, what is the final price of a DVD player that is marked \$79.98?
- **Solution:** Multiply 79.98 by $1 + .065 = 1.065$:

$$1.065 \times 79.98 = 85.18$$

- (You usually round up.)
- The restaurant tax in DeKalb is 9%. How much will a \$4.29 extra value meal cost?
- **Solution:** Multiply 4.29 by 1.09: $1.09 \times 4.29 = 4.68$

7. Tips

- If you wish to leave a 15% tip on a \$36 meal, how much is the tip?
- **Solution:** Multiply 36 by $.15 = 5.40$:
- **Rule of Thumb:** You can calculate 10% by just moving the decimal point.
  - Ten percent of \$36 is \$3.60.
  - To find twenty percent, just double this:
  - $2 \times 3.60 = 7.20$
  - To find 15 percent, add 10 percent plus half of 10 percent:

$$3.60 + \frac{1}{2}3.60 = 3.60 + 1.80 = 5.40$$

8. Eating Out

- If you plan on spending \$40 for two meals at a restaurant, and you need to add on 9% sales tax and a 15% tip, how much money do you need to bring on your date?
- **Solution:** Multiply 40 by both (1.15) and (1.09):

$$(1.15)(1.09)40 = 50.14$$

9. Investment and Interest

- You invest \$1000 in the bank at an annual interest rate of 4.5%. How much will you have in your account after one year?
- **Solution:** Multiply 1000 by 1.045: $1.045 \times 1000 = 1045$
- At 4.5% annual interest, how much will you have in your account after 3 years?
- Can you just add 3 times \$45 to the principal?
- **Answer:** It depends on whether the bank is offering simple interest or compound interest.

10. Two Types of Interest

Simple interest is computed on the original principal only.
At an annual rate of 4.5%, the accrued simple interest on a principal of \$1000 after three years will be $3 \times 45 = 135$.
With compound interest, the bank pays you interest on the interest you made in the first and second years.

11. Simple Interest

The formula for simple interest is very simple.
Variables:
r = annual interest rate
t = number of years
P = original principal
A = amount accumulated after t years

$$\boxed{A = P(1 + rt)}$$

12. Compound Interest

- At 4.5% annual interest, how much will you have in your account after 3 years?
- Assume the interest is compounded every year.
- You need to multiply 1000 by 1.045 three times, that is, multiply by $(1.045)^3$:
- $(1.045)^3 \times 1000 = 1141.17$
- Since $3 \times 45 = 135$, the interest on the interest added just \$6.17.

13. Compound Interest

The formula for compound interest is also simple.
Variables:
r = annual interest rate
t = number of years
P = original principal
A = amount accumulated after t years

$$\boxed{A = P(1 + r)^t}$$

14. More Compound Interest

- At 4.5% annual interest, how much will you have in your account after 30 years?
- **Solution:** Multiply 1000 by $(1.045)^{30}$:
- $A = (1.045)^{30} \times 1000 = 3745.32$
- Since $30 \times 45 = 1350$, the principal you would have without compounding the interest would be 2350 instead of 3745.
- You made \$1395 because the interest was compounded, more than the \$1350 in simple interest.

15. Moral

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- Compound Interest — Good!
- Simple Interest — Bad!

16. Monthly Compound Interest

- A bank pays you 6% annual interest, *compounded monthly.* What does this mean?
- **Answer:** They break up the year into 12 monthly periods, and pay you $6/12 = \frac{1}{2}$ percent interest for each period.
- Your account after 1 year is worth

$$(1.005)^{12} \times P = 1.0617 \times P,$$

where P is the beginning principal.

17. Computing Interest Rates

Your credit card company charges you \$4 in one month for a \$200 balance. What is your annual interest rate?
Answer: Break up the year into 12 months. Your interest rate is

$$\frac{\text{interest}}{\text{principal}} = \frac{\$4}{\$200} \cdot \frac{1}{\text{month}} \ \frac{12 \text{ months}}{1 \text{ year}} = 0.24$$

for an annual interest of 24%

18. Compound Interest Formula

The formula for compound interest with periodic interest conversions per year is:
Variables:
r = annual interest rate
m = number of interest periods per year
t = number of years
P = original principal
A = amount accumulated after t years

$$\boxed{A = P\left(1 + \frac{r}{m}\right)^{mt}}$$

19. Simplified Formula

The formula $A = P\left(1 + \frac{r}{m}\right)^{mt}$ can be written

$$\boxed{A = P(1+i)^n}$$

where $i = \frac{r}{m}$ and $n = mt$
Here i is the interest rate per period and n is the number of periods.

20. Monthly versus Yearly

- At 4.5% annual interest, the amount accrued in your account after 30 years is

yearly $A = (1.045)^{30} \times 1000 = 3745.32$

- What if the interst was compounded monthly?
- The monthly interst rate is $i = \frac{.045}{12} = .00375$
- The number of months is $n = 12 \times 30 = 360$

monthly $A = (1.00375)^{360} \times 1000 = 3847.70$

- It's worth a little more than \$102 to compound monthly versus annually.

21. A.P.R. versus A.P.Y.

- A.P.R. – annual percentage rate
- A.P.Y. – annual percentage yield
- sometimes called the effective rate
- For the previous problem, the A.P.R. is the advertised yearly rate of 6 percent.
- For the previous problem, the A.P.Y. is the actual yearly rate that accrues over the year.
- In this case, we calculated $(1.005)^{12} = 1.0617$
- So the A.P.Y. is 6.17 percent.
- Compounding the interest every month has the same effect as giving an interest rate of 6.17% at the end of the year.

22. The National Debt

- As of November, 2009 the national debt was 12 trillion dollars.
- In real numbers this is

$$\boxed{12{,}000{,}000{,}000{,}000}$$

- This figure is the money the U.S. government has had to borrow over the years because it has spent more than it has collected in taxes.
- The U.S. government borrows money by selling bonds–mostly treasury bonds, treasury bills, and savings bonds–to anyone who will buy them. (Foreign investors account for 25%.) In return for lending Uncle Sam the money, the bondholders are promised interest on the loan.

23. Interest on National Debt

- In 2008 the interest on the national debt was about

$$\boxed{451{,}154{,}000{,}000}$$

- The U.S. government paid more for interest on the national debt than it paid for defense, education, welfare, or any thing else.
- Current Monthly Interest: $\$36,513,000,000$

24. What each U.S. Citizen Owes

- Your Personal Share of the Interest in the National Debt is more than $\$84,462$
- Your share of the annual interest amounts to $261.46 per month.
- Interest figures can fluctuate widely from month to month.
- They are currently very low for the size of the debt, because recent, massive, short-term federal borrowing has been at near zero interest.
- This will change dramatically when the economic recovery begins.
- People and institutions that have parked money in US Treasury instruments will begin to move it into higher yield investments.

25. Going Both Ways

- A gas station increases prices by 10% because of the increase in fuel prices due to a gas shortage.
- After a while, the shortage is over, and the station decreases prices by 10%.
- Are the prices back to the original prices?
- **Answer:** Surprisingly, no!
  - Suppose the original price is \$1.00.
  - Ten percent of \$1.00 is .10
  - Increase \$1.00 by 10%: \$1.10
  - Ten percent of \$1.10 is .11
  - Decrease \$1.10 by 10%: $1.10 - .11 = .99$
- What happened?

26. Mathematical Explanation

- Decreasing by 10% means multiplying by $1 - .1 = .9$
- Increasing by 10% means multiplying by $1 + .1 = 1.1$
- But 1.1 times .9 does not result in 1.
- The correct product is
- $1.1 \times .9 = .99$

27. Investment Application

Aggressive versus Safe

- Dave and Donna both have one thousand dollars to invest over the next three years.
- Dave invests his money in a conservative money market account, which yields a steady 7 percent interest per year.
- Donna invests her money in more volatile "technology stocks," which earn her 30 percent interest in year 1 and year 2, but then lose 30 percent in the third year.
- Who has earned more after 3 years, Dave or Donna?

28. Dave's Investment

- **Answer:** Let's start with Dave first. His money has increased by a factor of $1.07^3 = 1.225$
- earning him 22.5 percent interest over 3 years.
- Question: 3 times 7 percent is 21 percent, so where does the extra 1.5 percent interest come from?
- Now consider Donna.
- You might think that the 30 percent loss in year 3 cancels the 30 percent increase in year 2, leaving her roughly the 30 percent increase she made in the first year.
- But it doesn't work like this.

29. Donna's Investment

- Her factor of increase is $1.30 \times 1.30 \times 0.70 = 1.183$, earning her 18.3 percent interest, far short of the 30% she made in year one.
- The flaw is that a 30% decrease does not cancel a 30% increase:
- do the math: $1.30 \times 0.70 = .91$,
- resulting in a net loss of 9%.
- Moral: bad years hurt more than good years help.

30. Inflation

- Assuming 3 percent inflation over the next 20 years, how much will a 6000 motorcycle cost twenty years from now?
- Solution: $(1.03)^{20} \times 6000 = 10,837$
- At 5 percent inflation, the value increases to $(1.05)^{20} \times 6000 = 15,920$
- If you invest money in a 3 percent certificate of deposit, and the rate of inflation is greater than 3 percent, then you are losing money (in terms of buying power).

31. Going in Reverse

- Stereo speakers are marked "40% off"
- If the sale price is \$100, what is the regular price of the speakers?
- Can you just add 40% of \$100, to compensate for the 40% discount?
  - Since 40% of \$100 is \$40, this method says the original price was \$140.
  - Let's check. What is 40% off \$140?
  - Answer: $.6 \times 140 = 84$, which is not \$100.
  - So the original price was **not** \$140.

32. Going in Reverse

- Stereo speakers are marked "40% off" If the sale price is \$100, what is the regular price of the speakers?
- **Solution:** If P is the original price, then we multiply P by .6 to get the sale price
- $0.6 \times P = 100$
- To find P **we must divide:**
- $P = \frac{100}{.6} = 166.67$

33. Sales Tax Example

- Assuming sales tax is 6.5%, what was the original price of a car if the price including the sales tax came to \$13,312.50?
- **Solution:** If P is the original price, then we multiply P by 1.065 to get the final price including tax.
- $1.065 \times P = 13312.50$
- To find P we must divide:
- $P = \frac{13312.50}{1.065} = 12500$

34. Interest Example

- You invest a principal at an effective annual rate of 6% interest.
- If your account is worth \$2650 after one year, how much did you initially invest?
- **Solution:** If P is the original principal, then we multiply P by 1.06 to get the amount in the account after one year.
- $1.06 \times P = 2650$
- To find P we must divide:
- $P = \frac{2650}{1.06} = 2500$

35. Present versus Future Value

In the formula

$$A = P(1+i)^n$$

- P is called the present value
- A is called the future value

Sometimes you want to know how much to invest now to obtain an accumulated amount in the future.

We can express the present value P in terms of the future value A by dividing:

$$P = \frac{A}{(1+i)^n}$$

36. Compound Interest Example

- You invest a principal at 6% interest, compounded monthly.
- If your account is worth \$3482.30 after one year, how much did you initially invest?
- Solution: If P is the original principal, then we multiply P by $(1+\frac{.06}{12})^{12} = 1.005^{12}$ to get the amount in the account after one year.
- $1.005^{12} \times P = 3482.30$
- To find P we must divide:
- $P = \frac{3482.30}{1.005^{12}} = 3280$

37. Savings Bonds

- How much does a \$100 U.S. Savings Bond cost if it matures in five years at 4% interest, compounded annually?.
- **Solution:** If P is the original price of the bond, then we multiply P by 1.04^5 to get the value of the bond after five years, which is guaranteed to be one hundred dollars:
- $$1.04^5 \times P = 100$$
- To find P we must divide:
- $P = \frac{100}{1.04^5} = 82.19$
- The difference of \$17.81 is the interest you earn for purchasing the bond.

38. A Simple Error

- A student from a previous semester once made a simple error in this calculation.
- Instead of writing
- $P = \frac{100}{1.04^5} = 82.19$
- this student wrote
- $P = \frac{100}{0.04^5}$
- omitting the 1 in 1.04
- My calculator says
- $\frac{100}{0.04^5} = 976,562,500$
- Does this answer seem reasonable?