3. Saving Money Every Month

- Suppose you put $M$ dollars in the bank every month, for six months.
- If the bank pays 0.005 monthly interest (6% annual interest), your money grows like this:

<table>
<thead>
<tr>
<th>month</th>
<th>formula</th>
<th>calculated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.005^5M</td>
<td>1.0253M</td>
</tr>
<tr>
<td>2</td>
<td>1.005^4M</td>
<td>1.0202M</td>
</tr>
<tr>
<td>3</td>
<td>1.005^3M</td>
<td>1.0150M</td>
</tr>
<tr>
<td>4</td>
<td>1.005^2M</td>
<td>1.0100M</td>
</tr>
<tr>
<td>5</td>
<td>1.005^1M</td>
<td>1.0050M</td>
</tr>
<tr>
<td>6</td>
<td>M</td>
<td>M</td>
</tr>
<tr>
<td>Total</td>
<td>Sum</td>
<td>6.0755M</td>
</tr>
</tbody>
</table>

4. Final Balance

- After 6 months, you account will be worth

$$1.005^5M + 1.005^4M + 1.005^3M + 1.005^2M + 1.005M + M$$

- Factoring out an $M$, this sum is

$$1.005^5 + 1.005^4 + 1.005^3 + 1.005^2 + 1.005 + 1)M$$

- We want a formula for the sum inside the parentheses.

5. A Summation Formula

To compute $S = x^5 + x^4 + x^3 + x^2 + x^1 + 1$ use the following trick:

Multiply $S$ by $x$ then subtract $S$.

$$xS = x^6 + x^5 + x^4 + x^3 + x^2 + x^1$$

$$S = x^5 + x^4 + x^3 + x^2 + x^1 + 1$$

$$xS - S = x^6 - 1$$

All the other terms on the right cancel!

The rest is easy:

$$xS - S = x^6 - 1 \implies (x - 1)S = x^6 - 1 \implies S = \frac{x^6 - 1}{x - 1}$$

6. Using the Summation Formula

We saw that if you deposit $M$ dollars in the bank for six consecutive months, then the balance after six months is

$$S = (1.005^5 + 1.005^4 + 1.005^3 + 1.005^2 + 1.005 + 1)M$$

Plugging $x = 1.005$ into the Summation Formula

$$1 + x + x^2 + \cdots + x^5 = \frac{x^6 - 1}{x - 1}$$

gives

$$S = \frac{1.005^6 - 1}{1.005 - 1}M = \frac{1.005^6 - 1}{.005}M$$
7. Final Calculation

- If you deposit $M = 100$ dollars every month for six months, at a monthly rate of $i = .005$, you will have
  
  $$S = \frac{1.005^6 - 1}{.005} \times 100 = 607.55 \text{ dollars after 6 months.}$$

- The $7.55$ represents accumulated monthly interest.

9. Long Term Savings

- Suppose you saved $M = 100$ dollars every month, for 45 years, at a monthly rate of $i = .005$.

- Here
  
  - $M = 100$
  - $i = .005$
  - $n = 12 \times 45 = 540$

- After 45 years the account balance would be
  
  $$S = \frac{1.005^{540} - 1}{.005} \times 100 = 275,599 \text{ dollars.}$$

- Your monthly contributions were $540 \times 100 = 54,000$, so most of the growth in the account is due to accrued interest.

10. Saving for Retirement

- You and your sister have different ideas about saving for your retirement at age 65.

- At age 25 you start put aside $200$ every month into an IRA yielding 7 percent interest.

- After ten years, at age 35, you decide that family obligations require you to save the $200$ for college money for your three kids.

- So you stop putting money into the account, but let it continue to collect 7 percent interest until you retire at age 65.

11. Saving for Retirement Cont’d

- Your sister, on the other hand, waits until her 45th birthday to begin saving for her retirement.

- Like you, she has $200$ taken out of her paycheck every month into a 7% IRA account.

- At age 65, who has saved more money: you or your sister?

8. Monthly Savings Formula

$$S = \frac{(1 + i)^n - 1}{i} M$$

where

- $M$ = amount saved per month
- $S$ = ending balance
- $i = r/12$ = monthly interest rate
- $n$ = number of months

12. Your Sister

- Let’s look at your sister first, since her situation is easier to analyze.

- The monthly interest rate is $i = .07/12$ for $n = 240$ months.

- Since $M = 200$, the monthly savings formula gives an ending balance after 20 years of

  $$\frac{(1 + .07/12)^{240} - 1}{.07/12} \times 200 = 104,185$$

- more than doubling the $240 \times 200 = 48,000$ she has invested.
13. Your Turn

• You on the other hand have invested one half as much money as your sister: $120 \times 200 = 24,000$.
• After ten years, when you are 35, your ending balance is computed by the Monthly Savings Formula (with $i = .07/12$ and $n = 120$ months):
  
  $$S = \frac{(1 + .07/12)^{120} - 1}{.07/12} \times 200 = 34,616.96$$

• Now this money earns compound interest over the next 30 years or $30 \times 12 = 360$ months, so that at age 65 your IRA account will be worth
  $$S = \frac{(1 + .07/12)^{360} - 1}{.07/12} \times 200 = 34,616.96$$
• You made 2.7 times as much as your sister, although you only invested half as much money.

14. Advice

• Moral: How much you accumulate for retirement depends upon three things:
  – (i) when you start saving,
  – (ii) how much you manage to save, and
  – (iii) how much your investments return over the long run.
• Of the three, when you start saving turns out to be the most important.
• Moral: Invest when you are young!

15. Monthly Payment $M$ to obtain Balance $S$

$$M = \frac{i}{(1 + i)^n - 1} S$$

This formula comes from the Monthly Savings Formula

$$S = \frac{(1 + i)^n - 1}{i} M$$

16. Saving for College

• Kaylee’s parents want to put aside money every month so that their daughter will have $25,000 for college when she turns 18.
• At 5% annual interest, how much do they need to save per month?
• Set the variables:
  – amount saved per month: $M$
  – desired ending balance: $25,000$
  – monthly interest rate: $i = .05/12$
  
  $$i = .004166667$$
  – number of months: $12 \times 18 = 216$

• $M = \frac{.004166667}{(1.004166667)^{216} - 1} \times 25,000 = 71.60$