1. Math 210 Finite Mathematics

Chapter 5.2 and 5.3
Annuities Mortgages Amortization
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2. Math 210 FINAL EXAM

Monday, May 4, 2015
12 - 1:50 pm
Cole Hall 100

3. Saving Money Every Month

- Suppose you put $M$ dollars in the bank every month, for six months.
- If the bank pays 0.005 monthly interest (6% annual interest), your money grows like this:

<table>
<thead>
<tr>
<th>month</th>
<th>formula</th>
<th>calculated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.005^5M</td>
<td>1.0253M</td>
</tr>
<tr>
<td>2</td>
<td>1.005^4M</td>
<td>1.0202M</td>
</tr>
<tr>
<td>3</td>
<td>1.005^3M</td>
<td>1.0150M</td>
</tr>
<tr>
<td>4</td>
<td>1.005^2M</td>
<td>1.0100M</td>
</tr>
<tr>
<td>5</td>
<td>1.005^1M</td>
<td>1.0050M</td>
</tr>
<tr>
<td>6</td>
<td>M</td>
<td>M</td>
</tr>
<tr>
<td>Total</td>
<td>Sum</td>
<td>6.0755M</td>
</tr>
</tbody>
</table>

4. Final Balance

- After 6 months, you account will be worth

$$1.005^5M + 1.005^4M + 1.005^3M + 1.005^2M + 1.005^1M + M$$

- Factoring out an $M$, this sum is

$$(1.005^5 + 1.005^4 + 1.005^3 + 1.005^2 + 1.005^1 + 1)M$$

- We want a formula for the sum inside the parentheses.

5. A Summation Formula

To compute $S = x^5 + x^4 + x^3 + x^2 + x + 1$ use the following trick:

Multiply $S$ by $x$ then subtract $S$.

$$xS = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

$$S = x^5 + x^4 + x^3 + x^2 + x + 1$$

$$xS - S = x^6 - 1$$

All the other terms on the right cancel!

The rest is easy:

$$xS - S = x^6 - 1 \implies (x - 1)S = x^6 - 1$$

$$\implies S = \frac{x^6 - 1}{x - 1}$$

6. Using the Summation Formula

We saw that if you deposit $M$ dollars in the bank for six consecutive months, then the balance after six months is

$$S = (1.005^5 + 1.005^4 + 1.005^3 + 1.005^2 + 1.005^1 + 1)M$$

Plugging $x = 1.005$ into the Summation Formula

$$1 + x + x^2 + \cdots + x^5 = \frac{x^6 - 1}{x - 1}$$

gives

$$S = \frac{1.005^6 - 1}{1.005 - 1}M = \frac{1.005^6 - 1}{.005}M$$
7. **Final Calculation**

- If you deposit $M = 100$ dollars every month for six months, at a monthly rate of $i = .005$, you will have
  
  $S = \frac{1.005^6 - 1}{.005} = 607.55$ dollars after 6 months.

- The 7.55 represents accumulated monthly interest.

8. **Monthly Savings Formula**

   $S = \frac{(1 + i)^n - 1}{i} \cdot M$

   where
   
   - $M =$ amount saved per month
   - $S =$ ending balance
   - $i = r/12 =$ monthly interest rate
   - $n =$ number of months

9. **Long Term Savings**

- Suppose you saved $M = 100$ dollars every month, for 45 years, at a monthly rate of $i = .005$.

- Here
  
  - $M = 100$
  - $i = .005$
  - $n = 12 \times 45 = 540$

- After 45 years the account balance would be
  
  $S = \frac{1.005^{540} - 1}{.005} = 275,599$ dollars.

- Your monthly contributions were $540 \times 100 = 54,000$, so most of the growth in the account is due to accrued interest.

10. **Saving for Retirement**

- You and your sister have different ideas about saving for your retirement at age 65.

- At age 25 you start put aside 200 every month into an IRA yielding 7 percent interest.

- After ten years, at age 35, you decide that family obligations require you to save the $200 for college money for your three kids.

- So you stop putting money into the account, but let it continue to collect 7% interest until you retire at age 65.

11. **Saving for Retirement Cont’d**

- Your sister, on the other hand, waits until her 45th birthday to begin saving for her retirement.

- Like you, she has $200 taken out of her paycheck every month into a 7% IRA account.

- At age 65, who has saved more money: you or your sister?

12. **Your Sister**

- Let's look at your sister first, since her situation is easier to analyze.

- The monthly interest rate is $i = .07/12$ for $n = 240$ months.

- Since $M = 200$, the monthly savings formula gives an ending balance after 20 years of
  
  \[
  \frac{(1 + .07/12)^{240} - 1}{.07/12} \cdot 200 = 104,185
  \]

- more than doubling the $240 \times 200 = 48,000$ she has invested.
13. Your Turn

- You on the other hand have invested one half as much money as your sister: 
  \( 120 \times 200 = 24,000 \).
- After ten years, when you are 35, your ending balance is computed by the Monthly Savings Formula (with \( i = .07/12 \) and \( n = 120 \) months):
  \[
  S = \frac{(1 + .07/12)^{120} - 1}{.07/12} 200 = 34,616.96
  \]
- Now this money earns compound interest over the next 30 years or \( 30 \times 12 = 360 \) months, so that at age 65 your IRA account will be worth 
  \( (1 + .07/12)^{360} \times S = 280,968.48 \)
- You made 2.7 times as much as your sister, although you only invested half as much money.

15. Monthly Payment \( M \) to Obtain Balance \( S \)

\[
M = \frac{i}{(1+i)^n - 1} S
\]

This formula comes from the Monthly Savings Formula

\[
S = \frac{(1+i)^n - 1}{i} M
\]

14. Advice

- Moral: How much you accumulate for retirement depends upon three things:
  - (i) when you start saving,
  - (ii) how much you manage to save, and
  - (iii) how much your investments return over the long run.
- Of the three, when you start saving turns out to be the most important.
- Moral: Invest when you are young!

16. Saving for College

- Kaylee’s parents want to put aside money every month so that their daughter will have $25,000 for college when she turns 18.
- At 5% annual interest, how much do they need to save per month?
- Set the variables:
  - amount saved per month: \( M \)
  - desired ending balance: 25,000
  - monthly interest rate: \( i = .05/12 \)
  \[
  = .004166667
  \]
  - number of months: \( 12 \times 18 = 216 \)
- \[
M = \frac{.004166667}{(1.004166667)^{216} - 1} 25,000 = 71.60
\]
17. Annuities

A sequence of payments made at regular intervals is called an **annuity**.

Special types of annuities:

- **certain** annuity — the term is a finite time period
- **ordinary** annuity — each payment is made at the end of a payment period
- **simple** annuity — the payment period coincides with the interest conversion period

The annuities we consider are (i) certain; (ii) ordinary; (iii) simple; and (iv) the periodic payments are all the same size.

19. Present Value of an Annuity

The idea is simply to determine what principal $P$ would you need to start with in order to end up with the balance $S$ after $n$ interest periods, compounded per period at an annual rate $r$?

That is, we set the compound interest formula equal to the future value of an annuity formula:

$$ P(1 + i)^n = \frac{(1 + i)^n - 1}{i} M $$

$$ \implies P = \frac{(1 + i)^n - 1}{i} \frac{M}{i} $$

$$ \implies P = \frac{1 - (1 + i)^{-n}}{i} M $$

18. Future Value of an Annuity

$$ S = \frac{(1 + i)^n - 1}{i} M $$

where

- $M$ = amount paid per investment period
- $S$ = ending balance
- $i$ = periodic interest rate
- $n$ = number of periods

Note that the interest periods need not be months, though frequently they are.

20. Annuities

- If you ever inherit (or win from the lottery) a large sum of money, you may wish to set up an annuity, typically with an insurance company or financial institution.
- An annuity works exactly like a home mortgage, reversing the roles of lender and borrower.
- With a home mortgage, a financial institution gives you a large sum of money (to buy your house) in exchange for your monthly payments over the next $n$ months.
- With an annuity, you give the financial institution a large sum of money and they agree to pay you the monthly (or annual) payments over a certain time period.
21. The Lottery

- Typically lottery winnings are always paid as annuities these days.
- Suppose you win one million dollars in a contest.
- The rules of the contest state that you will be paid 40,000 per year over the next 25 years.
- If the interest rate is \( r = .065 \), how much does the company sponsoring the contest need to pay to set up the annuity?

**Solution:** Compute the present value of an annuity which pays 40,000 per year at a rate .065 over 25 years.

- Note that payments are annual, not monthly.

22. Lottery Continued

- Variables in the Present Value Formula
  - annual interest rate: \( i = r = .065 \)
  - number of payments (in years): \( n = 25 \)
  - annual payment: \( M = 40,000 \)

- Present Value Calculation
  \[
P = \frac{1 - (1 + i)^{-n}}{i} \times M
\]

- So it costs the company less than 488 thousand dollars to give away a million dollars.

23. Amortization of Loans

Suppose you purchase a home or car by borrowing a principal of \( P \) dollars. Then \( P \) may be thought of as the present value of an annuity and the value of \( M \) in the formula

\[
P = \frac{1 - (1 + i)^{-n}}{i} \times M
\]

is the amount you need to pay every month in order to pay off the loan in \( n \) payments. Solving for \( M \) in terms of \( P \):

\[
M = \frac{i}{1 - (1 + i)^{-n}} \times P
\]

24. Monthly Payback formula

\[
M = \frac{i}{1 - (1 + i)^{-n}} \times P
\]

where

- \( M = \) monthly payment
- \( P = \) principal debt
- \( i = r/12 = \) monthly interest rate
- \( n = \) number of months

25. Calculator Tips

- You need to use parentheses around \( 1 + i \)
- You need parentheses around the entire denominator
- You need to use the negative button (not subtract) on the exponent \(-n\)
- You can store \( i = r/12 \) in the memory of your calculator
- The Monthly Payback Equation then becomes

\[
\text{RCL MEM ÷ \left(1 - (1 + \text{RCL MEM})^{-n}\right) \times P}
\]

26. Amortization

To amortize a loan means to set aside money regularly for future payment of the debt. The general formula for the amortization of a loan is

\[
M = \frac{i}{1 - (1 + i)^{-n}} \times P
\]

where

- \( M = \) payment per period
- \( P = \) principal debt
- \( i = \) periodic interest rate
- \( = r/\)number of periods per year
- \( n = \) number of periods

Typically, the periods are months.
27. **Buying a Car**

- You want to buy a Dodge Neon for $12,500.
- Which is the better deal:
  - $1000 cash back and a 3 year bank loan at 8.5% interest or
  - the promotional 1.9% loan from Dodge for the entire $12,500?

28. **Option 1. Cash Back**

- For the cash back option, you need to borrow
  - principal borrowed: \( P = 11,500 \)
  - annual interest rate: \( r = .085 \)
  - monthly interest rate: \( i = r/12 \)
  - number of months: \( n = 12 \times 3 = 36 \)
- The monthly payment for this option is
  \[
  M = \frac{.007083333}{1 - 1.007083333^{-36}}11500 = 363.03
  \]

29. **Option 2. Low Interest**

- For the 1.9% interest option, you need to borrow
  - principal borrowed: \( P = 12,500 \)
  - annual interest rate: \( r = .019 \)
  - monthly interest rate: \( i = r/12 \)
    \[
    = .001583333
    \]
  - number of months: \( n = 12 \times 3 = 36 \)
- The monthly payment for this option is
  \[
  M = \frac{.001583333}{1 - 1.001583333^{-36}}12500 = 357.49
  \]

30. **Comparison**

- Option 1. Payment is $363.03
- Option 2. Payment is $357.49
- **Conclusion:** Option 2 (with the lower interest) is slightly better.
- It saves about $5.54 per month.
- Over 36 months, this savings amounts to almost $200.

31. **Principal in Terms of Monthly Payment**

\[
P = \frac{1 - (1 + i)^{-n}}{i} M
\]

This is just the formula for the present value of an annuity.

32. **Buying a House**

- You wish to buy a house on a 30 year fixed mortgage.
- The interest rate is a fixed 8% (per year).
- The largest mortgage you can afford is $1000 per month.
- What is the largest principal you can afford if the interest rate is
  - 8 percent
  - 4.25 percent
33. Buying a House — High Interest

• Variables:
  - annual interest rate : \( r = 0.08 \)
  - monthly interest rate : \( i = \frac{r}{12} \)
    \[ = 0.0066666667 \]
  - number of months : \( n = 12 \times 30 = 360 \)
  - monthly payment : \( M = 1000 \)

• Largest Principal You Can Borrow

\[
P = \frac{1 - (1.0066667)^{-360}}{0.0066667} \times 1000 = 136,283.50
\]

34. Buying a House — Low Interest

• Variables:
  - annual interest rate : \( r = 0.0425 \)
  - monthly interest rate : \( i = \frac{r}{12} \)
    \[ = 0.00354166667 \]
  - number of months : \( n = 12 \times 30 = 360 \)
  - monthly payment : \( M = 1000 \)

• Largest Principal You Can Borrow

\[
P = \frac{1 - (1.0066667)^{-360}}{0.0066667} \times 1000 = 203,276.86
\]

35. Buying a House Comparison

<table>
<thead>
<tr>
<th>Interest</th>
<th>Principal</th>
</tr>
</thead>
<tbody>
<tr>
<td>8%</td>
<td>136,283.50</td>
</tr>
<tr>
<td>4.25%</td>
<td>203,276.86</td>
</tr>
</tbody>
</table>

Note: the amount that you will pay — one thousand dollars per month for 30 years — is the same for both cases.

36. Interest Paid 30 - year loan

<table>
<thead>
<tr>
<th>year</th>
<th>principal</th>
<th>interest</th>
<th>payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>first</td>
<td>91.44</td>
<td>908.56</td>
<td>136283.50</td>
</tr>
<tr>
<td>1</td>
<td>99.03</td>
<td>900.97</td>
<td>135145.04</td>
</tr>
<tr>
<td>2</td>
<td>107.25</td>
<td>892.75</td>
<td>133912.08</td>
</tr>
<tr>
<td>5</td>
<td>136.24</td>
<td>863.76</td>
<td>129564.53</td>
</tr>
<tr>
<td>10</td>
<td>202.97</td>
<td>797.03</td>
<td>119554.30</td>
</tr>
<tr>
<td>15</td>
<td>302.40</td>
<td>697.60</td>
<td>104640.61</td>
</tr>
<tr>
<td>20</td>
<td>450.52</td>
<td>549.48</td>
<td>82421.51</td>
</tr>
<tr>
<td>21 4mo</td>
<td>501.06</td>
<td>498.94</td>
<td>74841.26</td>
</tr>
<tr>
<td>25</td>
<td>671.21</td>
<td>328.79</td>
<td>49318.48</td>
</tr>
<tr>
<td>29</td>
<td>923.36</td>
<td>76.64</td>
<td>11495.84</td>
</tr>
<tr>
<td>last</td>
<td>993.38</td>
<td>6.62</td>
<td>993.44</td>
</tr>
</tbody>
</table>

Total Interest = 360 \( M - P \) = 360 \times 1000 - 136,283.50 = 223,716.50

37. Reducing the Interest

• Total Interest = 360 \( M - P \)
  \[ = 360 \times 1000 - 136,283.50 = 223,716.50 \]

• If you paid the same principal 136,283.50 off in a 15 year loan at the same interest rate 8%, what would the monthly payments be?
  \[ i = \frac{0.08}{12} \]
  \[ n = 180 \]
  \[ P = 136,283.50 \]

\[
M = \frac{0.0066667}{1 - (1.0066667)^{-180}} \times 136,283.50 = 1302.40
\]

38. Interest Paid 15-year loan

<table>
<thead>
<tr>
<th>year</th>
<th>principal</th>
<th>interest</th>
<th>payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>first</td>
<td>393.84</td>
<td>908.56</td>
<td>136283.50</td>
</tr>
<tr>
<td>1</td>
<td>426.53</td>
<td>875.87</td>
<td>133880.18</td>
</tr>
<tr>
<td>2</td>
<td>461.93</td>
<td>840.47</td>
<td>126069.89</td>
</tr>
<tr>
<td>5</td>
<td>586.77</td>
<td>715.63</td>
<td>107345.13</td>
</tr>
<tr>
<td>6 4mo</td>
<td>652.56</td>
<td>649.82</td>
<td>97472.54</td>
</tr>
<tr>
<td>10</td>
<td>874.19</td>
<td>428.21</td>
<td>64231.42</td>
</tr>
<tr>
<td>14</td>
<td>1202.59</td>
<td>99.81</td>
<td>14970.86</td>
</tr>
<tr>
<td>last</td>
<td>1293.38</td>
<td>8.62</td>
<td>1292.44</td>
</tr>
</tbody>
</table>

Total Interest = 180 \( M - P \) = 180 \times 1000 - 136,283.50 = 98,147.80
39. **Comparison**

- **30 - year loan**
  - Monthly Payment : \( M = 1000 \)
  - Number of months : \( n = 360 \)
  - Total paid : 360,000
  - Total interest : 223,716.50

- **15 - year loan**
  - Monthly Payment : \( M = 1302.40 \)
  - Number of months : \( n = 180 \)
  - Total paid : 234,432
  - Total interest : 98,148

40. **Advice**

- The 15-year loan saves over $125,000 in interest.
- Moral: Go for the shorter loan if you can afford it.

41. **A Calculator Error**

- If $750 is the largest monthly house payment you can afford, what is the largest principle you can borrow from the bank over 15 years at 7.4% interest?
- The Maximum Principle Formula says
  \[
  P = \frac{1 - (1 + .074/12)^{-180}}{.074/12} \times 750
  \]
- A student from a previous semester typed
  \[
  (1 - (1 + .074 ÷ 12) \land -180) ÷ .074 ÷ 12 \times 750 =
  \]
- and got the answer $565.30
- What went wrong?
- Does this answer seem reasonable?

42. **Credit Cards Advantages**

- convenient
- necessary for auto rental
- phone orders
- records and receipts
- eliminates worry of cash

43. **Credit Cards Disadvantages**

- usually high interest rate
- encourages excess spending
- balance can keep growing
- large proportion spent on interest
- “a continued life of debt”

44. **Credit Card Interest**

- You owe $2000 on your Visa Card
- The interest rate is 18%
- You are required to pay 2% of the balance per month
- The monthly interest is \( \frac{18}{12} = .015 \)
- On a balance of $2000, you will pay
  \[
  0.015 \times 2000 = 30
  \]
- Your minimal monthly payment: $40.00
45. Credit Card Interest Cont’d
• The interest you pay: $30.00
• Amount paid on balance: $10.00
• Getting an A in this class: priceless
• Notice that 3/4 of your payment is going directly to interest.
• At this rate, you will pay $8000 for the $2000 worth of merchandise you bought.

46. How to get out of credit card debt
• You owe $2000 to Visa.
• The interest rate is 18% or 1.5% per month.
• You would like to pay this off in two years.
• First, cut up the card with scissors or at least promise not to use it for two years.
• Now use the Monthly Payback Formula:
  principal borrowed : \( P = 2,000 \)
  annual interest rate : \( r = .18 \)
  monthly interest rate : \( i = r/12 = .015 \)
  number of months : \( n = 12 \times 2 = 24 \)

47. Getting out of credit card debt
• Calculate the monthly payment:
  \[ M = \frac{.015}{1 - 1.015^{-24}} 2000 = 99.85 \]
• Send Visa a check for $99.85 every month for two years and you will be out of debt.
• Total Payments: 24 \times 99.85 = 2396.40
• Total Interest: 2396.40 - 20000 = 396.40