# 1. Math 210 Finite Mathematics

Chapter 5.2 and 5.3
Annuities Mortgages Amortization

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Math 210 Website: http://math.niu.edu/courses/math210

# 2. Math 210 FINAL EXAM

Monday, December 7, 2015
2 - 3:50 pm
Cole Hall 100

## 3. Saving Money Every Month

- Suppose you put $M$ dollars in the bank every month, for six months.
- If the bank pays 0.005 monthly interest (6% annual interest), your money grows like this:

| month | formula | calculated |
|---|---|---|
| 1 | $1.005^5 M$ | $1.0253M$ |
| 2 | $1.005^4 M$ | $1.0202M$ |
| 3 | $1.005^3 M$ | $1.0150M$ |
| 4 | $1.005^2 M$ | $1.0100M$ |
| 5 | $1.005^1 M$ | $1.0050M$ |
| 6 | $M$ | $M$ |
| Total | Sum | $6.0755M$ |

## 4. Final Balance

- After 6 months, you account will be worth

$$1.005^5 M + 1.005^4 M + 1.005^3 M + 1.005^2 M + 1.005^1 M + M$$

- Factoring out an $M$, this sum is
- $(1.005^5 + 1.005^4 + 1.005^3 + 1.005^2 + 1.005^1 + 1)M$
- We want a formula for the sum inside the parentheses.

## 5. A Summation Formula

To compute $S = x^5 + x^4 + x^3 + x^2 + x^1 + 1$
use the following trick:
Multiply $S$ by $x$ then subtract $S$.

$$xS = x^6 + x^5 + x^4 + x^3 + x^2 + x^1$$

$$S = \quad x^5 + x^4 + x^3 + x^2 + x^1 + 1$$

$$xS - S = x^6 - 1$$

All the other terms on the right cancel!
The rest is easy:
$xS - S = x^6 - 1 \implies (x-1)S = x^6 - 1$
$\implies S = \frac{x^6-1}{x-1}$

## 6. Using the Summation Formula

We saw that if you deposit $M$ dollars in the bank for six consecutive months, then the balance after six months is
$S = (1.005^5+1.005^4+1.005^3+1.005^2+1.005^1+1)M$
Plugging $x = 1.005$ into the Summation Formula

$$\boxed{1 + x + x^2 + \cdots + x^5 = \frac{x^6-1}{x-1}}$$

gives

$$S = \frac{1.005^6 - 1}{1.005 - 1}M = \frac{1.005^6 - 1}{.005}M$$

## 7. Final Calculation

- If you deposit $M = 100$ dollars every month for six months, at a monthly rate of $i = .005$, you will have
- $S = \dfrac{1.005^6 - 1}{.005}100 = 607.55$ dollars after 6 months.
- The 7.55 represents accumulated monthly interest.

## 8. Monthly Savings formula

$$\boxed{S = \frac{(1+i)^n - 1}{i} M}$$

where

- $M$ = amount saved per month
- $S$ = ending balance
- $i = r/12$ = monthly interest rate
- $n$ = number of months

## 9. Long Term Savings

- Suppose you saved $M = 100$ dollars every month, for 45 years, at a monthly rate of $i = .005$.
- Here
  - $M = 100$
  - $i = .005$
  - $n = 12 \times 45 = 540$
- After 45 years the account balance would be
- $S = \dfrac{1.005^{540} - 1}{.005}100 = 275,599$ dollars.
- Your monthly contributions were $540 \times 100 = 54,000$, so most of the growth in the account is due to accrued interest.

## 10. Saving for Retirement

- You and your sister have different ideas about saving for your retirement at age 65.
- At age 25 you start put aside 200 every month into an IRA yielding 7 percent interest.
- After ten years, at age 35, you decide that family obligations require you to save the $200 for college money for your three kids.
- So you stop putting money into the account, but let it continue to collect 7% interest until you retire at age 65.

## 11. Saving for Retirement Cont'd

- Your sister, on the other hand, waits until her 45th birthday to begin saving for her retirement.
- Like you, she has $200 taken out of her paycheck every month into a 7% IRA account.
- At age 65, who has saved more money: you or your sister?

## 12. Your Sister

- Let's look at your sister first, since her situation is easier to analyze.
- The monthly interest rate is $i = .07/12$ for $n = 240$ months.
- Since $M = 200$, the monthly savings formula gives an ending balance after 20 years of
- $\dfrac{(1+.07/12)^{240} - 1}{.07/12}200 = 104,185$
- more than doubling the $240 \times 200 = 48,000$ she has invested.

## 13. Your Turn

- You on the other hand have invested one half as much money as your sister: $120 \times 200 = 24,000$.
- After ten years, when you are 35, your ending balance is computed by the Monthly Savings Formula (with $i = .07/12$ and $n = 120$ months):
- $S = \dfrac{(1 + .07/12)^{120} - 1}{.07/12} 200 = 34,616.96$
- Now this money earns compound interest over the next 30 years or $30 \times 12 = 360$ months, so that at age 65 your IRA account will be worth
- $(1 + .07/12)^{360} \times S = 280,968.48$
- You made 2.7 times as much as your sister, although you only invested half as much money.

## 14. Advice

- Moral: How much you accumulate for retirement depends upon three things:
  - (i) when you start saving,
  - (ii) how much you manage to save, and
  - (iii) how much your investments return over the long run.
- Of the three, when you start saving turns out to be the most important.
- Moral: Invest when you are young!

## 15. Monthly Payment $M$ to obtain Balance $S$

$$\boxed{M = \frac{i}{(1+i)^n - 1} S}$$

This formula comes from the Monthly Savings Formula

$$\boxed{S = \frac{(1+i)^n - 1}{i} M}$$

## 16. Saving for College

- Kaylee's parents want to put aside money every month so that their daughter will have \$25,000 for college when she turns 18.
- At 5% annual interest, how much do they need to save per month?
- Set the variables:
  - amount saved per month: $M$
  - desired ending balance: $25,000$
  - monthly interest rate: $i = .05/12$
    $= .004166667$
  - number of months: $12 \times 18 = 216$
- $M = \dfrac{.004166667}{(1.004166667)^{216} - 1} 25,000 = 71.60$

## 17. Annuities

A sequence of payments made at regular intervals is called an annuity.
Special types of annuities:

- certain annuity — the term is a finite time period
- ordinary annuity — eah payment is made at the end of a payment period
- simple annuity — the payment period coincides with the interest conversion period

The annuities we consider are (i) certain; (ii) ordinary; (iii) simple; and (iv) the periodic payments are all the same size.

## 18. Future Value of an Annuity

$$\boxed{S = \frac{(1+i)^n - 1}{i} M}$$

where

- $M$ = amount paid per investment period
- $S$ = ending balance
- $i$ = periodic interest rate
- $n$ = number of periods

Note that the interest periods need not be months, though frequently they are.

## 19. Present Value of an Annuity

The idea is simply to determine what principal $P$ would you need to start with in order to end up with the balance $S$ after $n$ interest periods, compounded per period at an annual rate $r$? That is, we set the compund interest formula equal to the future value of an annuity formula:

$$P(1+i)^n = \frac{(1+i)^n - 1}{i} M$$

$$\Longrightarrow \quad P = \frac{(1+i)^n - 1}{(1+i)^n\, i} M$$

$$\Longrightarrow \quad \boxed{P = \frac{1-(1+i)^{-n}}{i} M}$$

## 20. Annuities

- If you ever inherit (or win from the lottery) a large sum of money, you may wish to set up an annuity, typically with an insurance company or financial institution.
- An annuity works exactly like a home mortgage, reversing the roles of lender and borrower.
- With a home mortgage, a financial institution gives you a large sum of money (to buy your house) in exchange for your monthly payments over the next $n$ months.
- With an annuity, you give the financial institution a large sum of money and they agree to pay you the monthly (or annual) payments over a certain time period.

## 21. The Lottery

- Typically lottery winnings are always paid as annuities these days.
- Suppose you win one million dollars in a contest.
- The rules of the contest state that you will be paid $40,000$ per year over the next 25 years.
- If the interest rate is $r = .065$, how much does the company sponsoring the contest need to pay to set up the annuity?
- **Solution:** Compute the present value of an annuity which pays $40,000$ per year at a rate .065 over 25 years.
- Note that payments are annual, not monthly.

## 22. Lottery Continued

- Variables in the Present Value Formula

$$\begin{aligned} \text{annual interest rate}: &\quad i = r = .065 \\ \text{number of payments (in years)}: &\quad n = 25 \\ \text{annual payment}: &\quad M = 40,000 \end{aligned}$$

- Present Value Calculation $P = \dfrac{1-(1.065)^{-25}}{.065} 40000 = 487,915.07$
- So it costs the company less than 488 thousand dollars to give away a million dollars.

## 23. Amortization of Loans

Suppose you purchase a home or car by borrowing a principal of $P$ dollars.
Then $P$ may be thought of as the present value of an annuity and the value of $M$ in the formula
$P = \dfrac{1-(1+i)^{-n}}{i} M$
is the amount you need to pay every month in order to pay off the loan in $n$ payments.
Solving for $M$ in terms of $P$:
$M = \dfrac{i}{1-(1+i)^{-n}} P$

## 24. Monthly Payback formula

$$\boxed{M = \frac{i}{1-(1+i)^{-n}} P}$$

where

- $M$ = monthly payment
- $P$ = principal
- $i = r/12$ = monthly interest rate
- $n$ = number of months

## 25. Calculator Tips

- You need to use parentheses around $1+i$
- You need parentheses around the entire denominator
- You need to use the negative button (not subtract) on the exponent $-n$
- You can store $i = r/12$ in the memory of your calculator
- The Monthly Payback Equation then becomes

$$\text{RCL MEM} \div \Big(1 - \big(1 + \text{RCL MEM}\big) \wedge -n\Big) \times P$$

## 26. Amortization

To amortize a loan means to set aside money regularly for future payment of the debt.
The general formula for the amortization of a loan is

$$\boxed{M = \frac{i}{1-(1+i)^{-n}} P}$$

where

- $M$ = payment per period
- $P$ = principal debt
- $i$ = periodic interest rate
  = $r$/number of periods per year
- $n$ = number of periods

Typically, the periods are months.

## 27. Buying a Car

- You want to buy a Dodge Neon for \$12,500.
- Which is the better deal:
  - \$1000 cash back and a 3 year bank loan at 8.5% interest or
  - the promotional 1.9% loan from Dodge for the entire \$12,500?

## 28. Option 1. Cash Back

- For the cash back option, you need to borrow
  - principal borrowed : $P = 11,500$
  - annual interest rate : $r = .085$
  - monthly interest rate : $i = r/12$

    $= .007083333$
  - number of months : $n = 12 \times 3 = 36$
- The monthly payment for this option is $M = \frac{.007083333}{1 - 1.00708333^{-36}} 11500 = 363.03$

## 29. Option 2. Low Interest

- For the 1.9% interest option, you need to borrow
  - principal borrowed : $P = 12,500$
  - annual interest rate : $r = .019$
  - monthly interest rate : $i = r/12$

    $= .001583333$
  - number of months : $n = 12 \times 3 = 36$
- The monthly payment for this option is
- $M = \frac{.001583333}{1 - 1.00158333^{-36}} 12500 = 357.49$

## 30. Comparison

- Option 1. Payment is \$363.03
- Option 2. Payment is \$357.49
- **Conclusion:** Option 2 (with the lower interest) is slightly better.
- It saves about \$5.54 per month.
- Over 36 months, this savings amounts to almost \$200.

## 31. Principal in terms of Monthly Payment

$$\boxed{P = \frac{1 - (1 + i)^{-n}}{i} M}$$

This is just the formula for the present value of an annuity.

## 32. Buying a House

- You wish to buy a house on a 30 year fixed mortgage.
- The interest rate is a fixed 8% (per year).
- The largest mortgage you can afford is \$1000 per month.
- What is the largest principal you can afford if the interest rate is
  - 8 percent
  - 4.25 percent

33. Buying a House — High Interest

- Variables:
  - annual interest rate : $r = .08$
  - monthly interest rate : $i = r/12$
    $= .0066666667$
  - number of months : $n = 12 \times 30 = 360$
  - monthly payment : $M = 1000$
- Largest Principal You Can Borrow

$$P = \frac{1-(1.0066667)^{-360}}{.0066667}1000$$
$$= 136,283.50$$

34. Buying a House — Low Interest

- Variables:
  - annual interest rate : $r = .0425$
  - monthly interest rate : $i = r/12$
    $= .00354166667$
  - number of months : $n = 12 \times 30 = 360$
  - monthly payment : $M = 1000$
- Largest Principal You Can Borrow

$$P = \frac{1-(1.0066667)^{-360}}{.0066667}1000$$
$$= 203,276.86$$

35. Buying a House Comparison

| Interest | Principal |
|---|---|
| 8% | 136,283.50 |
| 4.25% | 203,276.86 |

Note: the amount that you will pay — one thousand dollars per month for 30 years — is the same for both cases.

36. Interest Paid 30 - year loan

| year | principal | interest | payoff |
|---|---|---|---|
| first | 91.44 | 908.56 | 136283.50 |
| 1 | 99.03 | 900.97 | 135145.04 |
| 2 | 107.25 | 892.75 | 133912.08 |
| 5 | 136.24 | 863.76 | 129564.53 |
| 10 | 202.97 | 797.03 | 119554.30 |
| 15 | 302.40 | 697.60 | 104640.61 |
| 20 | 450.52 | 549.48 | 82421.51 |
| 21 4mo | 501.06 | 498.94 | 74841.26 |
| 25 | 671.21 | 328.79 | 49318.48 |
| 29 | 923.36 | 76.64 | 11495.84 |
| last | 993.38 | 6.62 | 993.44 |

37. Reducing the Interest

- Total Interest $= 360M - P$
  $= 360 \times 1000 - 136,283.50$
  $= 223,716.50$
- If you paid the same principal 136,283.50 off in a 15 year loan at the same interest rate 8%, what would the monthly payments be?
- $i = .08/12 \quad n = 180 \quad P = 136,283.50$
- $$M = \frac{.0066667}{1-(1.0066667)^{-180}} \cdot 136,283.50$$
  $= 1302.40$

38. Interest Paid 15-year loan

| year | principal | interest | pay off |
|---|---|---|---|
| first | 393.84 | 908.56 | 136283.50 |
| 1 | 426.53 | 875.87 | 131380.18 |
| 2 | 461.93 | 840.47 | 126069.89 |
| 5 | 586.77 | 715.63 | 107345.13 |
| 6 4mo | 652.58 | 649.82 | 97472.54 |
| 10 | 874.19 | 428.21 | 64231.42 |
| 14 | 1202.59 | 99.81 | 14970.86 |
| last | 1293.78 | 8.62 | 1292.44 |

Total Interest $= 180M - P = 180 \times 1000 - 136,283.50 = 98,147.80$

## 39. Comparison

- **30 - year loan**

$$\begin{aligned} \text{Monthly Payment}: &\quad M = 1000 \\ \text{Number of months}: &\quad n = 360 \\ \text{Total paid}: &\quad 360,000 \\ \text{Total interest}: &\quad 223,716.50 \end{aligned}$$

- **15 - year loan**

$$\begin{aligned} \text{Monthly Payment}: &\quad M = 1302.40 \\ \text{Number of months}: &\quad n = 180 \\ \text{Total paid}: &\quad 234,432 \\ \text{Total interest}: &\quad 98,148 \end{aligned}$$

## 40. Advice

- The 15-year loan saves over $125,000 in interest.
- Moral: Go for the shorter loan if you can afford it.

## 41. A Calculator Error

- If $750 is the largest monthly house payment you can afford, what is the largest principle you can borrow from the bank over 15 years at 7.4% interest?
- The Maximum Principle Formula says
- $P = \dfrac{1 - \left(1 + .074/12)\right)^{-180}}{.074/12} \times 750$
- A student from a previous semester typed
- $(1 - (1 + .074 \div 12) \wedge -180) \ \div .074 \div 12 \times 750 =$
- and got the answer $565.30
- What went wrong?
- Does this answer seem reasonable?

## 42. Credit Cards Advantages

- convenient
- necessary for auto rental
- phone orders
- records and receipts
- eliminates worry of cash

## 43. Credit Cards Disadvantages

- usually high interest rate
- encourages excess spending
- balance can keep growing
- large proportion spent on interest
- "a continued life of debt"

## 44. Credit Card Interest

- You owe $2000 on your Visa Card
- The interest rate is 18%
- You are required to pay 2% of the balance per month
- The monthly interest is $\dfrac{.18}{12} = .015$
- On a balance of $2000, you will pay

$$0.015 \times 2000 = 30$$

- Your minimal monthly payment: $ 40.00

45. Credit Card Interest Cont'd

- The interest you pay: \$ 30.00
- Amount paid on balance: \$ 10.00
- Getting an A in this class: priceless
- Notice that 3/4 of your payment is going directly to interest.
- At this rate, you will pay \$8000 for the \$2000 worth of merchandise you bought.

46. How to get out of credit card debt

- You owe \$2000 to Visa.
- The interest rate is 18% or 1.5% per month.
- You would like to pay this off in two years.
- First, cut up the card with scissors or at least promise not to use it for two years.
- Now use the Monthly Payback Formula:

$$\begin{aligned} \text{principal borrowed}: &\quad P = 2,000 \\ \text{annual interest rate}: &\quad r = .18 \\ \text{monthly interest rate}: &\quad i = r/12 = .015 \\ \text{number of months}: &\quad n = 12 \times 2 = 24 \end{aligned}$$

47. Getting out of credit card debt

- Calculate the monthly payment:
- $M = \dfrac{.015}{1 - 1.015^{-24}} 2000 = 99.85$
- Send Visa a check for \$99.85 every month for two years and you will be out of debt.
- Total Payments: $24 \times 99.85 = 2396.40$
- Total Interest: $2396.40 - 20000 = 396.40$