1. Math 210 Finite Mathematics

- Chapter 3.1 Halfplanes
- Chapter 3.2 Linear Programming Problems
- Chapter 3.3 Graphical Solution
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- Math 210 Website: http://math.niu.edu/courses/math210

2. Solving Inequalities

You solve inequalities the way you solve equations:

Algebra Rule	Equation	Inequality
	$2x - 5 = 3$	$2x - 5 \leq 3$
Add 5 to both sides	$2x = 8$	$2x \leq 8$
Divide by 2	$x = 4$	$x \leq 4$

With One Important Difference...

3. The Inequality Trap

Consider the following

Algebra Rule	Equation	Inequality
	$-3x + 2 = -4$	$-3x + 2 \geq -4$
Subtract 2	$-3x = -6$	$-3x \geq -6$
Divide by -3	$x = 2$	$x \leq 2$

Why did the inequality change from

"greater than" ($\geq$) to "less than" ($\leq$)?

4. The Negative Rule for Inequalities

When multiplying or dividing an inequality by a negative number, the direction of the inequality changes.

- Multiply $2 < 3$ by -7:
- Solution: $-14 > -21$
- 14 below zero is warmer than 21 below zero
- Divide $-3x \geq -6$ by -3:
- Solution: $x \leq 2$
- Check. Try value of x less than 2:
  - $x = 1$ satisfies $-3(1) \geq -6$
  - $x = 0$ satisfies $-3(0) \geq -6$
  - $x = -2$ satisfies $-3(-2) \geq -6$

5. Graphing a Line in Standard Form

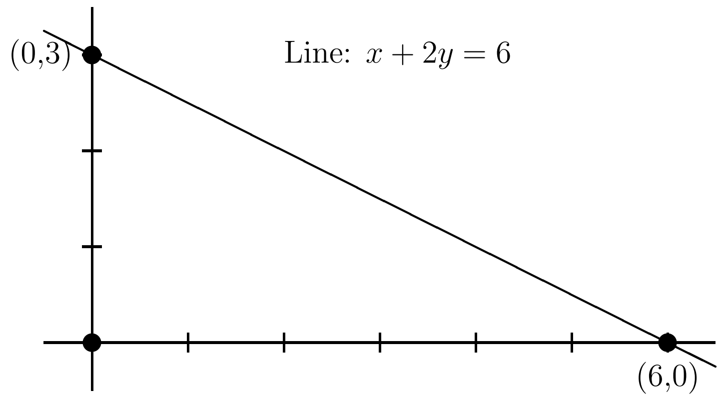
Graph the line $x + 2y = 6$

- Step a. Find x–intercept.
  - When $y = 0$,
  - $x + 0 = 6 \implies x = 6$
  - So x–intercept is $(6, 0)$
- Step b. Find y–intercept.
  - When $x = 0$,
  - $0 + 2y = 6 \implies y = 3$
  - So y–intercept is $(0, 3)$

6. Plot and Graph

- Step c. Draw line.
  - Mark the points $(6, 0)$ and $(0, 3)$ on your graph paper.
  - Using a ruler, draw a line through these two points.
  - This is the line $x + 2y = 6$.

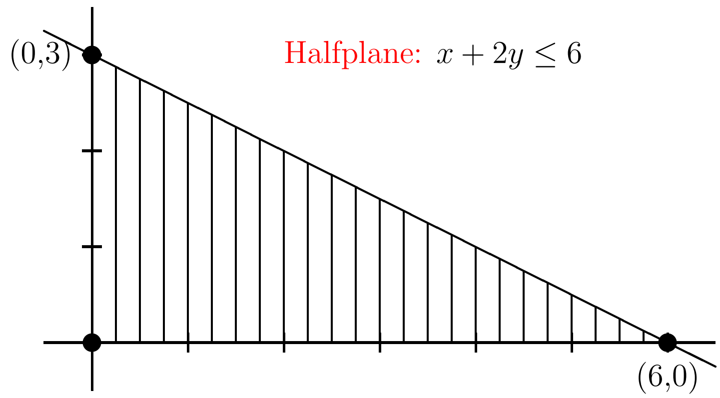
7. Graph of the Line



8. Graphing an Inequality

- Graph the inequality $x + 2y \leq 6$
- **Fact:** This inequality is a halfplane, whose edge is the line $x + 2y = 6$, which we just graphed.
- **Idea:** To determine which halfplane it is (above or below the line), try a test point, usually $(0, 0)$.
  - Is $0 + 0 \leq 6$?
  - Yes. So the halfplane $x + 2y \leq 6$ contains $(0, 0)$ and therefore lies below the line $x + 2y = 6$.

9. Graph of the Halfplane



10. Systems of Inequalities

Sketch the region determined by the following set of inequalities:

- $x + y \leq 16$
- $5x + 2y \leq 50$
- $y \leq 12$
- $x \geq 0,\ y \geq 0$

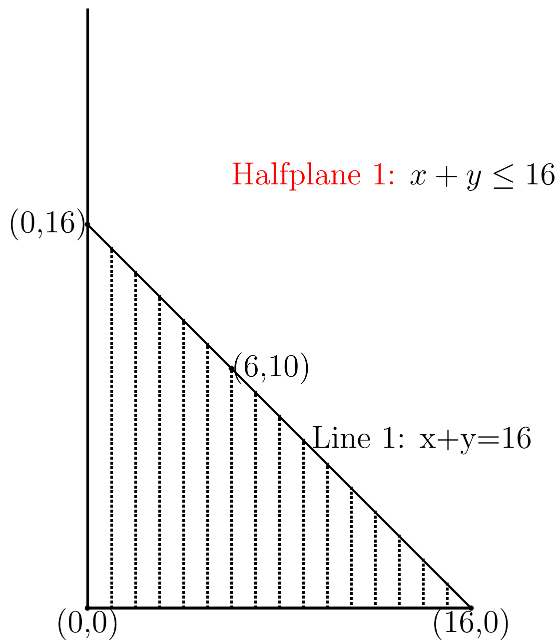
11. Halfplane 1: $x + y \leq 16$

- Step a. Find x–intercept.
  - When $y = 0$,
  - $x + 0 = 16 \implies x = 16$
  - So x–intercept is $(16, 0)$
- Step b. Find y–intercept.
  - When $x = 0$,
  - $0 + y = 16 \implies y = 16$
  - So y–intercept is $(0, 16)$

12. Halfplane 1: $x + y \leq 16$

- Step c. Draw the line
  - Mark the points $(16, 0)$ and $(0, 16)$ on your graph paper.
  - Using a ruler, draw a line through these two points.
  - This is Line 1.
- Step d. Use $(0, 0)$ as a test point to identify the halfplane.
  - Is $0 + 0 \leq 16$?
  - Yes. So the halfplane $x + y \leq 16$ contains $(0, 0)$ and therefore lies below the line $x + y = 16$.

13. Graph of Halfplane 1



14. Halfplane 2: $5x + 2y \leq 50$

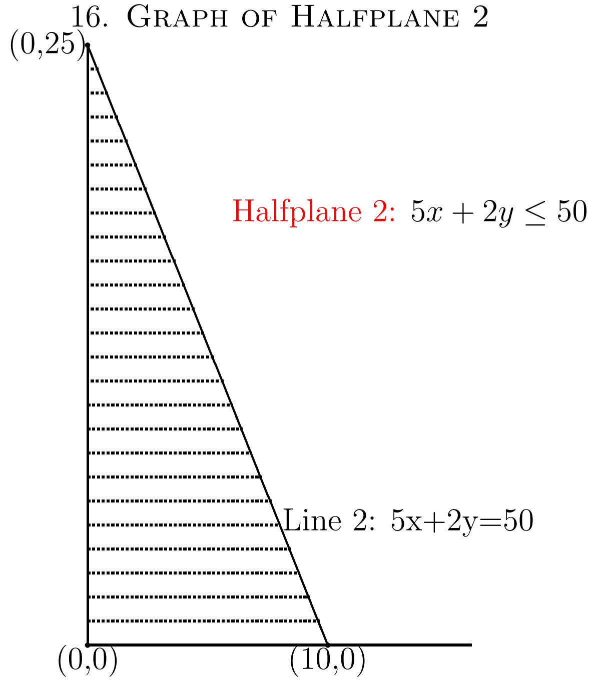
- Step a. Find x–intercept.
  - When $y = 0$,
  - $5x + 0 = 50 \implies x = 10$
  - So x–intercept is $(10, 0)$
- Step b. Find y–intercept.
  - When $x = 0$,
  - $0 + 2y = 50 \implies y = 25$

– So y–intercept is $(0, 25)$

15. Halfplane 2: $5x + 2y \leq 50$

- Step c. Draw line
  - Mark the points $(10, 0)$ and $(0, 25)$ on your graph paper.
  - Using a ruler, draw a line through these two points.
  - This is Line 2.
- Step d. Use $(0, 0)$ as a test point to identify the halfplane.
  - Is $0 + 0 \leq 50$?
  - Yes. So the halfplane $5x + 2y \leq 50$ contains $(0, 0)$ and therefore lies below the line $5x + 2y = 50$.

16. Graph of Halfplane 2



17. Where do Line 1 and Line 2 intersect?

- Step a. Write the equations:

  (1) $5x + 2y = 50$
- (2) $x + y = 16$

- Step b. Solve for one of the variables.
  original(1) $5x + 2y = 50$
- $2 \times (2)$ $2x + 2y = 32$
  Subtract $3x \quad = 18$
- So $x = \frac{18}{3} = 6$

18. Interesection Cont'd

- Step c. Solve for the other variable.
- Plug $x = 6$ back into original equation
  (2) $x + y = 16$
  $6 + y = 16$
- implies $y = 10$
- Step d. Write down the solution:
- The intersection point is $(6, 10)$.

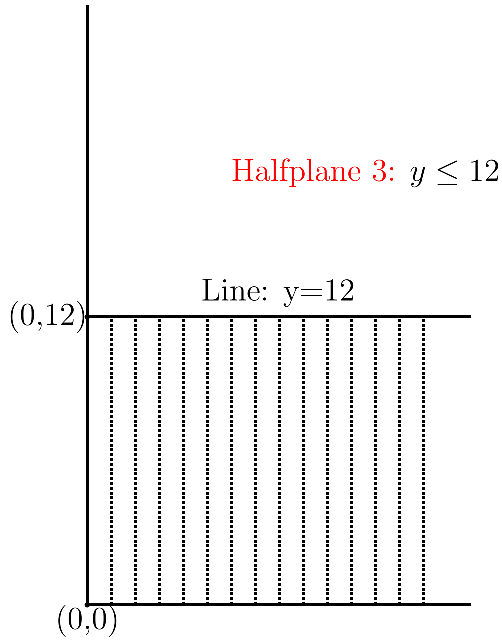
19. Halfplane 3: $y \leq 12$

- Step a. Find y–intercept.
  - When $x = 0$,
  - $y = 12$
  - So y–intercept is $(0, 12)$
- Step b. There is no x–intercept, since the horizontal line is parallel to the x–axis.
- Step c. Draw the line
  - Mark the point $(0, 12)$ on your graph paper.
  - Using a ruler, draw a horizontal line through this point.
  - This is Line 3.

20. Halfplane 3: $y \leq 12$

- Step d. Use $(0, 0)$ as a test point to identify the halfplane.
  - Is $0 \leq 12$?
  - Yes. So the halfplane $y \leq 12$ contains $(0, 0)$ and therefore lies below the line $y = 12$.

21. Graph of Halfplane 3



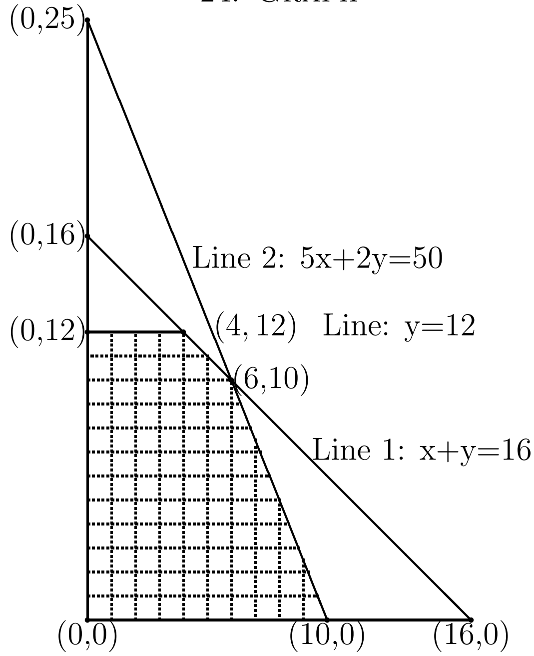
22. Where do Line 1 and Line 3 intersect?

- Step a. Write the equations:
- (1) $x + y = 16$
  (2) $y = 12$
- Step b. Plug $y = 12$ into equation (1)
- $x + 12 = 16$
  $\implies x = 4$
- Step c. Write down the solution:
- The intersection point is $(4, 12)$.

23. Putting it all together

- Step a. Graph Line 1, using the points $(16, 0)$ and $(0, 16)$
- Step b. Graph Line 2, using the points $(10, 0)$ and $(0, 25)$
- Step c. Graph Line 3, the horizontal line through the point $(0, 12)$
- Step d. Label the corner points $(6, 10)$, $(4, 12)$, $(0, 16)$, $(10, 0)$, and $(0, 0)$
- Step e. Shade in the intersection of the 3 halfplanes.

24. Graph



25. Linear Programming Problems

A linear programming problems has the form:

- **Maximize** an objective function
  - usually profit
  - or revenue
- **subject to** certain conditions
  - usually resource
  - or time constraints
- The objective function and the restraining conditions are all linear, that is no squares or worse.

26. Solving Linear Programming Problems

- The feasible region is the set of points satisfying the constraining equations.
- This region will consist of the intersection of halfplanes, whose respective lines meet in corner points of the region.

- When the feasible set is finite, the maximum (or minimum) is found by the following

Linear Programming Rule:

Maximums (and **minimums**) occur at corner points

27. Linear Programming Example

- So to find the maximum (or minimum) we simply evaluate the objective function at all the corner points of the feasible region.
- Maximize $P = 7x + 5y$ subject to the following set of inequalities:
  - $x + y \leq 16$
  - $5x + 2y \leq 50$
  - $y \leq 12$
  - $x \geq 0,\ y \geq 0$
- We have seen that the corner points are $(0, 0)$, $(0, 16)$, $(4, 12)$, $(6, 10)$, and $(10, 0)$

28. Solving the Problem

x	y	$P = 7x + 5y$
0	0	0
0	12	60
4	12	88
6	10	92
10	0	70

Maximum value of P is

- 92 and occurs at the point $(6, 10)$

Note: The minimum value is 0 at the point $(0, 0)$

29. EPA Regulations

A cement manufacture produces at least 3.2 million barrels of cement annually. The EPA (Environmental Protection Agency) finds that his operation emits 2.5 pounds of dust for each barrel produced. The EPA rules that annual emissions must be reduced to 1.8 million pounds. To do this,the manufactures plans to replace the present dust collectors with two types of electronic precipitators: Type I reduces emissions to .5 pounds per barrel and costs 16 cents per barrel; Type II reduces emissions to .3 pounds per barrel and costs 20 cents per barrel. The manufacturer does not want to spend more than .8 million dollars on the precipitation process. Graph the feasible set that meets the EPAs requirements?

30. Variables

Let x be the number of barrels of cement (in millions produced) using the first type of precipitator.

Let y be the number of barrels of cement (in millions produced) using the second type of precipitator.

31. Data Table

Type	Pounds per Barrel	Cost per Barrel
I	.5	$.16
II	.3	$.20

Total Barrels Produced: $P = x + y$

Total Emissions: $E = .5x + .3y$

Total Cost: $C = .16x + .20y$

32. Constraints

There are four kinds of constraints:

- need at least 3.2 million barrels per year
  Equivalently, $x + y \geq 3.2$
- EPA restriction: no more than 1.8 million pounds of emissions
  Equivalently, $.5x + .3y \leq 1.8$
- Cost factor: no more than .8 million dollars on total precipitation process

Equivalently, $.16x + .20y \leq 0.8$
- none of x, y can be negative

33. The Feasible Set

Based on the 4 types of constraints:

- $x + y \geq 3.2$
- $.5x + .3y \leq 1.8$
- $.16x + .2y \leq .8$
- $x, y \geq 0$

Multiply the second constraint by 10 and the third constraint by 5 to get:

- $x + y \geq 3.2$
- $5x + 3y \leq 18$
- $.8x + y \leq 4$
- $x, y \geq 0$

34. Halfplane $x + y \geq 3.2$

Line: $x + y = 3.2$

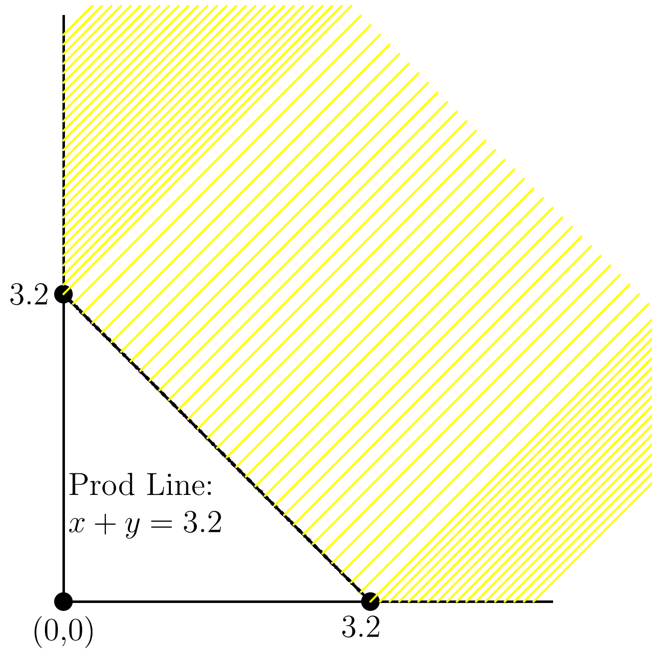
Intercepts: $(3.2, 0)$ and $(0, 3.2)$

Using the test point (0,0) we see

$0 + 0 \geq 3.2$ is false.

So the halfplane lies above the line $x + y = 3.2$

35. Halfplane $x + y \geq 3.2$



36. Halfplane $5x + 3y \leq 18$

Line: $5x + 3y = 18$

When $x = 0$, $3y = 18$, or $y = 6$

When $y = 0$, $5x = 18$, or $y = 3.6$

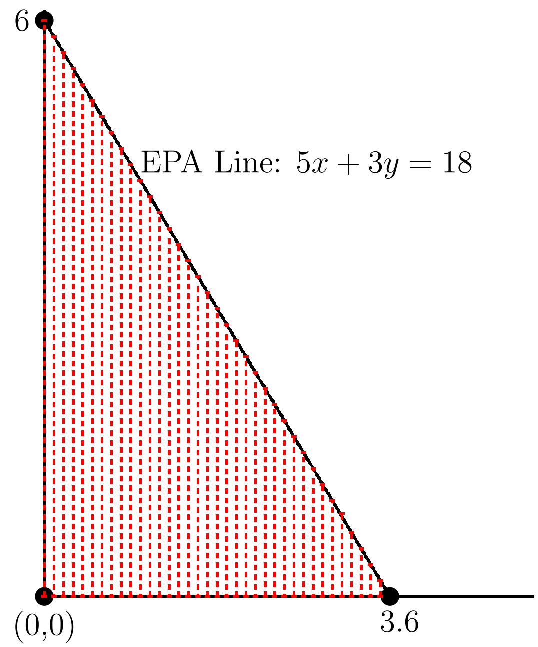
Intercepts: $(3.6, 0)$ and $(0, 6)$

Using the test point (0,0) we see

$0 + 0 \leq 18$ is true.

So the halfplane lies below the line $5x + 3y = 18$

37. HALFPLANE $5x + 3y \leq 18$



38. HALFPLANE $.8x + y \leq 4$

Line: $.8x + y = 4$

When $x = 0$, $y = 4$

When $y = 0$, $.8x = 4$, or $y = \frac{4}{.8} = 5$

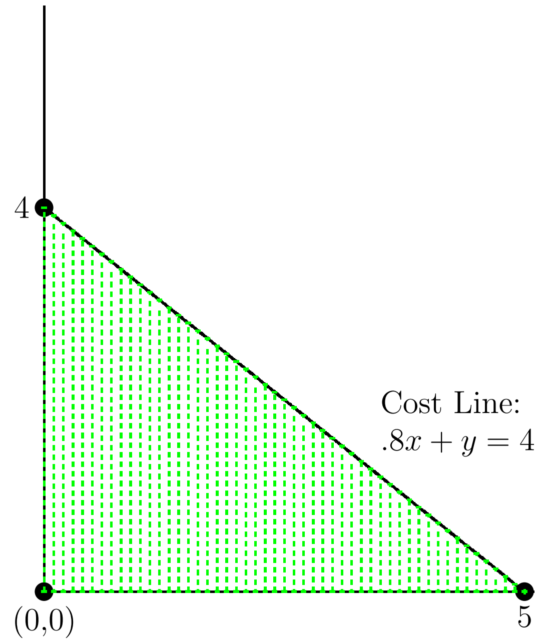
Intercepts: $(5, 0)$ and $(0, 4)$

Using the test point (0,0) we see

$0 + 0 \leq 4$ is true.

So the halfplane lies below the line $.8x + y = 4$

39. Halfplane $.8x + y \leq 4$



40. Where do Lines 2 and 3 intersect?

Line 2: $5x + 3y \leq 18$

Line 3: $.8x + y \leq 4$

Multiply line 3 by 3:

Line 2: $5x + 3y = 18$

$3\times$ Line 3: $2.4x + 3y = 12$

Subtract:

$$2.6x + 0y = 6$$

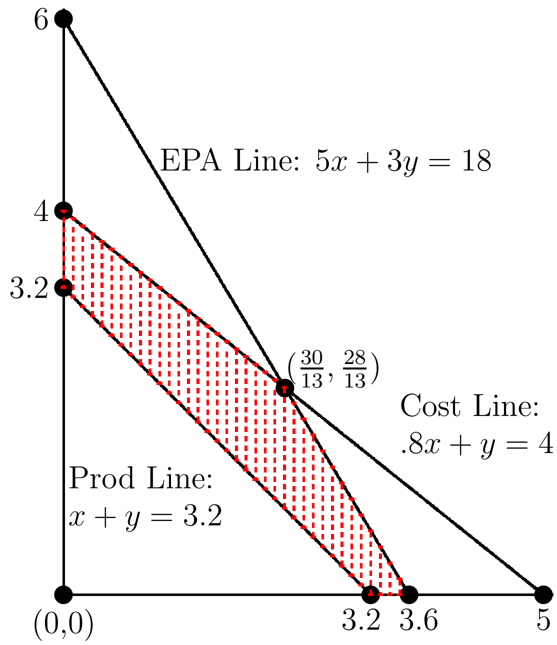
$x = \frac{6}{2.6} = \frac{3}{1.3} = \frac{30}{13}$

From Line 3,

$y = 4 - .8x = 4 - \frac{8}{10}\left(\frac{30}{13}\right) = \frac{52}{13} - \frac{24}{13} = \frac{28}{13}$

Intersection Point: $\left(\frac{28}{13}, \frac{30}{13}\right)$

41. Feasible Region



42. Construction Problem

A contractor builds two types of homes: the standard model and the deluxe model. The standard model requires one lot, \$12,000 capital, 150 labor-days to build, and is sold for a profit of \$2400. The deluxe model requires one lot, \$32,000 capital, 200 labor-days to build, and is sold for a profit of \$3400. The contractor has 150 lots. The bank is willing to loan him \$2,880,000 for the project and he has a maximum labor force available of 24,000 labor-days. How many houses should he build to realize the greatest profit?

43. Set the Variables

Let

- $x =$ the number of standard houses built
- $y =$ the number of deluxe houses built

The problem is to find the values of x and y which will

- maximize the profit and
- not exceed the resources of lots, capitol, or labor

44. Organizing the Data

Resource	Standard	Deluxe	Available
Lot	1	1	150
Capital	12,000	32,000	2,880,000
Labor	150	200	24,000
Profit	2400	3400	P

Resource Inequalities:

- Lot $1x + 1y \leq 150$
- Capital $12000x + 32000y \leq 2880000$
- Labor $150x + 200y \leq 24000$

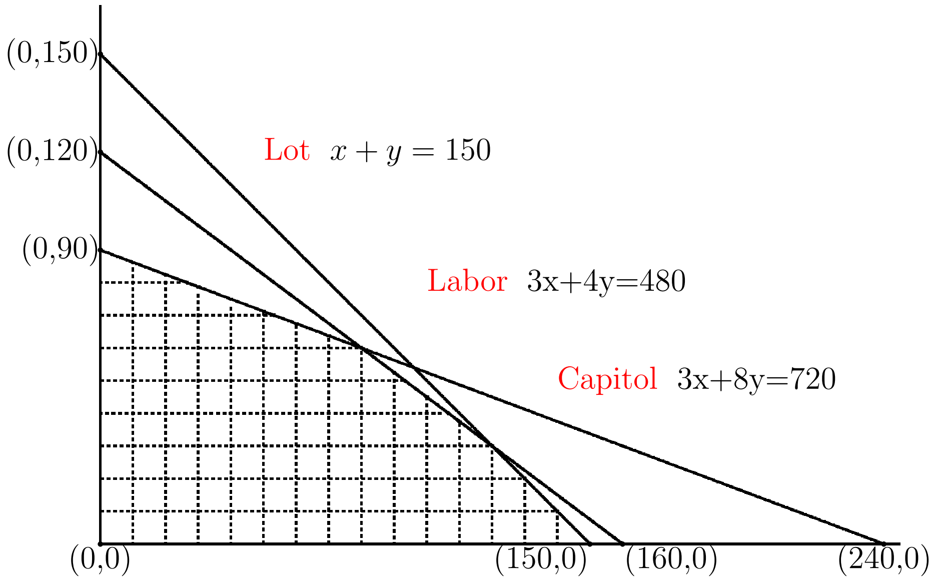
Profit Equation:

- Profit $P = 2400x + 3400y$

45. Simplify the Math

- Resource Inequalities and Profit:
  - Lot $1x + 1y \leq 150$
  - Capital $12000x + 32000y \leq 2880000$
  - Labor $150x + 200y \leq 24000$
  - Profit $P = 2400x + 3400y$
- Math Simplification
  - Divide Capital Inequality by 4000
  - Divide Labor Inequality by 50
- Simplified Equations:
  - Lot $x + y \leq 150$
  - Capital $3x + 8y \leq 720$
  - Labor $3x + 4y \leq 480$
  - Profit $P = 2400x + 3400y$

46. Graph



47. Where do Capitol and Labor Lines Cross?

- Write the equations:

Capitol	$3x + 8y = 720$
Labor	$3x + 4y = 480$
Subtract	$4y = 240$

- Divide by 4:
- So $y = \dfrac{240}{4} = 60$
- Plug $y = 60$ back into the Labor equation $3x + 4y = 480$

48. Capitol and Labor Line Intersection

$3x + 4 \cdot 60 = 480$

$3x + 240 = 480$

$3x = 480 - 240 = 240$

$x = \frac{240}{3} = 80$

- The intersection point is $(80, 60)$.

49. Where do Labor and Lot Lines Intersect?

- Write the equations:

  Labor $3x + 4y = 480$

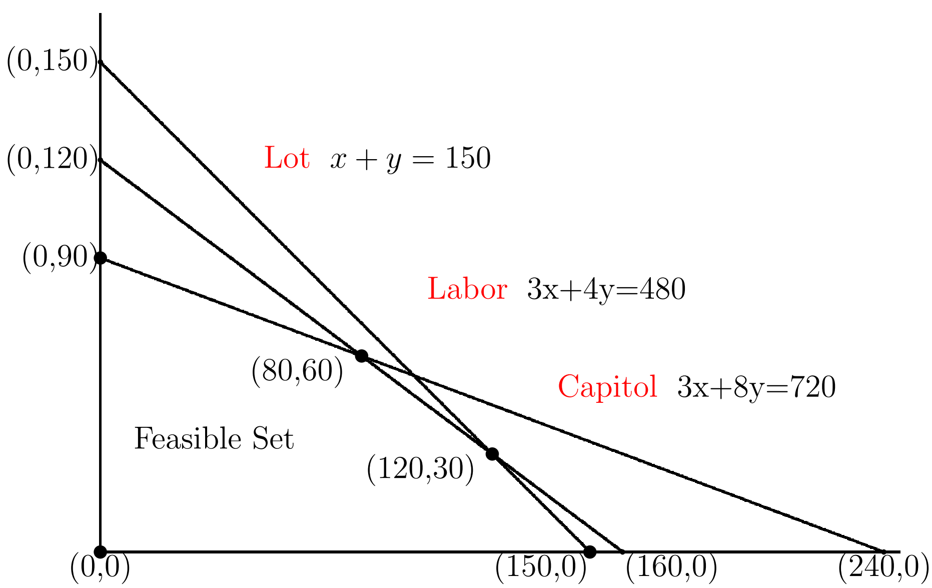
  Lot $x + y = 150$
- Multiply Lot equation by 3

  Capitol $3x + 4y = 480$

  Lot $3x + 3y = 450$

  Subtract $y = 30$
- Plug $y = 30$ back into the Lot equation
- $x + 30 = 150$ or $x = 120$
- The intersection point is $(120, 30)$

50. Graph



51. Solving the Problem

x	y	$P = 2400x + 3400y$
0	0	0
0	90	\$306,000
80	60	\$396,000
120	30	\$390,000
150	0	\$360,000

Maximum Profit

- 80 standard
- 60 deluxe
- \$396,000 profit
- Note: 10 lots are unused