1. Solving Inequalities
You solve inequalities the way you solve equations:

<table>
<thead>
<tr>
<th>Algebra Rule</th>
<th>Equation</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$2x - 5 = 3$</td>
<td>$2x - 5 \leq 3$</td>
</tr>
<tr>
<td>Add 5 to both sides</td>
<td>$2x = 8$</td>
<td>$2x \leq 8$</td>
</tr>
<tr>
<td>Divide by 2</td>
<td>$x = 4$</td>
<td>$x \leq 4$</td>
</tr>
</tbody>
</table>

With One Important Difference...

2. The Inequality Trap
Consider the following

<table>
<thead>
<tr>
<th>Algebra Rule</th>
<th>Equation</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$-3x + 2 = -4$</td>
<td>$-3x + 2 \geq -4$</td>
</tr>
<tr>
<td>Subtract 2</td>
<td>$-3x = -6$</td>
<td>$-3x \geq -6$</td>
</tr>
<tr>
<td>Divide by $-3$</td>
<td>$x = 2$</td>
<td>$x \leq 2$</td>
</tr>
</tbody>
</table>

Why did the inequality change from “greater than” ($\geq$) to “less than” ($\leq$)?

3. The Negative Rule for Inequalities
When multiplying or dividing an inequality by a negative number, the direction of the inequality changes.
- Multiply $2 < 3$ by $-7$: $-14 > -21$
- $14$ below zero is warmer than $21$ below zero
- Divide $-3x \geq -6$ by $-3$: $x \leq 2$
- Check. Try value of $x$ less than $2$:
  - $x = 1$ satisfies $-3(1) \geq -6$
  - $x = 0$ satisfies $-3(0) \geq -6$
  - $x = -2$ satisfies $-3(-2) \geq -6$

4. Graphing a Line in Standard Form
Graph the line $x + 2y = 6$
- Step a. Find $x$-intercept.
  - When $y = 0$, $x + 0 = 6 \implies x = 6$
  - So $x$-intercept is $(6,0)$
- Step b. Find $y$-intercept.
  - When $x = 0$, $0 + 2y = 6 \implies y = 3$
  - So $y$-intercept is $(0,3)$
5. Plot and Graph

- Step c. Draw line.
  - Mark the points (6, 0) and (0, 3) on your graph paper.
  - Using a ruler, draw a line through these two points.
  - This is the line $x + 2y = 6$.

6. Graph of the Line

7. Graphing an Inequality

- Graph the inequality $x + 2y \leq 6$
- **Fact:** This inequality is a halfplane, whose edge is the line $x + 2y = 6$, which we just graphed.
- **Idea:** To determine which halfplane it is (above or below the line), try a test point, usually (0, 0).
  - Is $0 + 0 \leq 6$?
  - Yes. So the halfplane $x + 2y \leq 6$ contains (0, 0) and therefore lies below the line $x + 2y = 6$.

8. Graph of the Halfplane

9. Systems of Inequalities

Sketch the region determined by the following set of inequalities:
- $x + y \leq 16$
- $5x + 2y \leq 50$
- $y \leq 12$
- $x \geq 0, y \geq 0$

10. Halfplane 1: $x + y \leq 16$

- **Step a.** Find $x$-intercept.
  - When $y = 0$,
    - $x + 0 = 16 \implies x = 16$
  - So $x$-intercept is (16, 0)
- **Step b.** Find $y$-intercept.
  - When $x = 0$,
    - $0 + y = 16 \implies y = 16$
    - So $y$-intercept is (0, 16)
11. **HALFPLANE 1: \( x + y \leq 16 \)**

- Step c. Draw the line
  - Mark the points \((16,0)\) and \((0,16)\) on your graph paper.
  - Using a ruler, draw a line through these two points.
  - This is Line 1.
- Step d. Use \((0,0)\) as a test point to identify the halfplane.
  - Is \(0 + 0 \leq 16\)?
  - Yes. So the halfplane \( x + y \leq 16 \) contains \((0,0)\) and therefore lies below the line \( x + y = 16 \).

12. **GRAPH OF HALFPLANE 1**

13. **HALFPLANE 2: \( 5x + 2y \leq 50 \)**

- Step a. Find \( x\)-intercept.
  - When \( y = 0 \),
  - \( 5x + 0 = 50 \implies x = 10 \)
  - So \( x\)-intercept is \((10,0)\)
- Step b. Find \( y\)-intercept.
  - When \( x = 0 \),
  - \( 0 + 2y = 50 \implies y = 25 \)
  - So \( y\)-intercept is \((0,25)\)

14. **GRAPH OF HALFPLANE 2**

15. **HALFPLANE 2: \( 5x + 2y \leq 50 \)**

- Step c. Draw line
  - Mark the points \((10,0)\) and \((0,25)\) on your graph paper.
  - Using a ruler, draw a line through these two points.
  - This is Line 2.
- Step d. Use \((0,0)\) as a test point to identify the halfplane.
  - Is \(0 + 0 \leq 50\)?
  - Yes. So the halfplane \( 5x + 2y \leq 50 \) contains \((0,0)\) and therefore lies below the line \( 5x + 2y = 50 \).

16. **WHERE DO LINE 1 AND LINE 2 INTERSECT?**

- Step a. Write the equations:
  \[(1) \quad 5x + 2y = 50\]
  \[(2) \quad x + y = 16\]
- Step b. Solve for one of the variables.
  - \(2 \times (2)\) \(2x + 2y = 32\)
  - Subtract \(3x = 18\)
  - So \( x = \frac{18}{3} = 6 \)
17. INTERSECTION CONT’D
- Step c. Solve for the other variable.
- Plug $x = 6$ back into original equation
  (2) $x + y = 16$
  $6 + y = 16$
- implies $y = 10$
- Step d. Write down the solution:
- The intersection point is $(6, 10)$.

18. HALFPLANE 3: $y \leq 12$
- Step a. Find $y$–intercept.
  - When $x = 0$,
  - $y = 12$
  - So $y$–intercept is $(0, 12)$
- Step b. There is no $x$–intercept, since
  the horizontal line is parallel to the $x$–axis.
- Step c. Draw the line
  - Mark the point $(0, 12)$ on your
  - Using a ruler, draw a horizontal line
  - This is Line 3.

19. HALFPLANE 3: $y \leq 12$
- Step d. Use $(0, 0)$ as a test point to
  identify the halfplane.
  - Is $0 \leq 12$?
  - Yes. So the halfplane $y \leq 12$ contains
  $(0, 0)$ and therefore lies below
  the line $y = 12$.

20. GRAPH OF HALFPLANE 3

21. WHERE DO LINE 1 AND LINE 3 INTERSECT?
- Step a. Write the equations:
  (1) $x + y = 16$
- (2) $y = 12$
- Step b. Plug $y = 12$ into equation (1)
  $x + 12 = 16$
- $\implies x = 4$
- Step c. Write down the solution:
- The intersection point is $(4, 12)$.

22. PUTTING IT ALL TOGETHER
- Step a. Graph Line 1, using the points
  $(16, 0)$ and $(0, 16)$
- Step b. Graph Line 2, using the points
  $(10, 0)$ and $(0, 25)$
- Step c. Graph Line 3, the horizontal
  line through the point $(0, 12)$
- Step d. Label the corner points $(6, 10),
  (4, 12), (0, 16), (10, 0)$, and $(0, 0)$
- Step e. Shade in the intersection of the
  3 halfplanes.
24. **Linear Programming Problems**

A *linear programming problem* has the form:
- **Maximize** an objective function
  - usually profit
  - or revenue
- **subject to** certain conditions
  - usually resource
  - or time constraints
- The objective function and the restraining conditions are all *linear*, that is no squares or worse.

26. **Linear Programming Example**

- So to find the maximum (or minimum) we simply evaluate the objective function at all the corner points of the feasible region.
- Maximize $P = 7x + 5y$ subject to the following set of inequalities:
  - $x + y \leq 16$
  - $5x + 2y \leq 50$
  - $y \leq 12$
  - $x \geq 0, y \geq 0$
- We have seen that the corner points are $(0, 0), (0, 16), (4, 12), (6, 10), \text{ and } (10, 0)$
27. Solving the Problem

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>P = 7x + 5y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>12</td>
<td>60</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>88</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>92</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>70</td>
</tr>
</tbody>
</table>

Maximum value of \( P \) is
- 92 and occurs at the point (6, 10)

Note: The minimum value is 0 at the point (0, 0)

29. Variables

Let \( x \) be the number of barrels of cement (in millions produced) using the first type of precipitator.

Let \( y \) be the number of barrels of cement (in millions produced) using the second type of precipitator.

31. Constraints

There are four kinds of constraints:
- Need at least 3.2 million barrels per year
  Equivalently, \( x + y \geq 3.2 \)
- EPA restriction: no more than 1.8 million pounds of emissions
  Equivalently, \(.5x + .3y \leq 1.8\)
- Cost factor: no more than .8 million dollars on total precipitation process
  Equivalently, \(.16x + .20y \leq 0.8\)
- None of \( x, y \) can be negative

30. Data Table

<table>
<thead>
<tr>
<th>Type</th>
<th>Pounds per Barrel</th>
<th>Cost per Barrel</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>.5</td>
<td>$.16</td>
</tr>
<tr>
<td>II</td>
<td>.3</td>
<td>$.20</td>
</tr>
</tbody>
</table>

Total Barrels Produced: \( P = x + y \)
Total Emissions: \( E = .5x + .3y \)
Total Cost: \( C = .16x + .20y \)

32. The Feasible Set

Based on the 4 types of constraints:
- \( x + y \geq 3.2 \)
- \(.5x + .3y \leq 1.8 \)
- \(.16x + .2y \leq .8 \)
- \( x, y \geq 0 \)

Multiply the second constraint by 10 and the third constraint by 5 to get:
- \( x + y \geq 3.2 \)
- \( 5x + 3y \leq 18 \)
- \(.8x + y \leq 4 \)
- \( x, y \geq 0 \)

28. EPA Regulations

A cement manufacturer produces at least 3.2 million barrels of cement annually. The EPA (Environmental Protection Agency) finds that his operation emits 2.5 pounds of dust for each barrel produced. The EPA rules that annual emissions must be reduced to 1.8 million pounds. To do this, the manufacturer plans to replace the present dust collectors with two types of electronic precipitators: Type I reduces emissions to .5 pounds per barrel and costs 16 cents per barrel; Type II reduces emissions to .3 pounds per barrel and costs 20 cents per barrel. The manufacturer does not want to spend more than .8 million dollars on the precipitation process. Graph the feasible set that meets the EPAs requirements?
33. **Halfplane** \( x + y \geq 3.2 \)

Line: \( x + y = 3.2 \)
Intercepts: \((3.2, 0)\) and \((0, 3.2)\)
Using the test point \((0,0)\) we see
\(0 + 0 \geq 3.2\) is false.
So the halfplane lies above the line \(x+y = 3.2\)

34. **Halfplane** \( x + y \geq 3.2 \)

35. **Halfplane** \( 5x + 3y \leq 18 \)

Line: \( 5x + 3y = 18 \)
When \( x = 0 \), \( 3y = 18 \), or \( y = 6 \)
When \( y = 0 \), \( 5x = 18 \), or \( x = 3.6 \)
Intercepts: \((3.6, 0)\) and \((0, 6)\)
Using the test point \((0,0)\) we see
\(0 + 0 \leq 18\) is true.
So the halfplane lies below the line \(5x + 3y = 18\)

36. **Halfplane** \( 5x + 3y \leq 18 \)

EPA Line: \( 5x + 3y = 18 \)
37. **Halfplane** \(0.8x + y \leq 4\)

Line: \(0.8x + y = 4\)
When \(x = 0\), \(y = 4\)
When \(y = 0\), \(0.8x = 4\), or \(x = \frac{4}{0.8} = 5\)
Intercepts: \((5, 0)\) and \((0, 4)\)
Using the test point \((0,0)\) we see
\(0 + 0 \leq 4\) is true.
So the halfplane lies below the line \(0.8x + y = 4\)

38. **Halfplane** \(0.8x + y \leq 4\)

39. **Where do Lines 2 and 3 intersect?**

Line 2: \(5x + 3y = 18\)
Line 3: \(0.8x + y = 4\)
Multiply line 3 by 3:

Line 2: \(5x + 3y = 18\)

3 \times Line 3: \(2.4x + 3y = 12\)
Subtract:

\[
2.6x + 0y = 6
\]
\[
x = \frac{6}{2.6} = \frac{3}{1.3} = \frac{30}{13}
\]
From Line 3,

\[
y = 4 - 0.8x = 4 - \frac{8}{13} \left( \frac{30}{13} \right) = \frac{52}{13} - \frac{24}{13} = \frac{28}{13}
\]
Intersection Point: \(\left( \frac{30}{13}, \frac{28}{13} \right)\)
41. Construction Problem

A contractor builds two types of homes: the standard model and the deluxe model. The standard model requires one lot, $12,000 capital, 150 labor-days to build, and is sold for a profit of $2400. The deluxe model requires one lot, $32,000 capital, 200 labor-days to build, and is sold for a profit of $3400. The contractor has 150 lots. The bank is willing to loan him $2,880,000 for the project and he has a maximum labor force available of 24,000 labor-days. How many houses should he build to realize the greatest profit?

42. Set the Variables

Let

- $x$ = the number of standard houses built
- $y$ = the number of deluxe houses built

The problem is to find the values of $x$ and $y$ which will

- maximize the profit and
- not exceed the resources of lots, capital, or labor

43. Organizing the Data

<table>
<thead>
<tr>
<th>Resource</th>
<th>Standard</th>
<th>Deluxe</th>
<th>Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lot</td>
<td>1</td>
<td>1</td>
<td>150</td>
</tr>
<tr>
<td>Capital</td>
<td>12,000</td>
<td>32,000</td>
<td>2,880,000</td>
</tr>
<tr>
<td>Labor</td>
<td>150</td>
<td>200</td>
<td>24,000</td>
</tr>
<tr>
<td>Profit</td>
<td>2400</td>
<td>3400</td>
<td>$P$</td>
</tr>
</tbody>
</table>

Resource Inequalities:

- Lot $1x + 1y \leq 150$
- Capital $12000x + 32000y \leq 2880000$
- Labor $150x + 200y \leq 24000$

Profit Equation:

- Profit $P = 2400x + 3400y$

44. Simplify the Math

- Resource Inequalities and Profit:
  - Lot $1x + 1y \leq 150$
  - Capital $12000x + 32000y \leq 2880000$
  - Labor $150x + 200y \leq 24000$
  - Profit $P = 2400x + 3400y$

- Math Simplification
  - Divide Capital Inequality by 4000
  - Divide Labor Inequality by 50

- Simplified Equations:
  - Lot $x + y \leq 150$
  - Capital $3x + 8y \leq 720$
  - Labor $3x + 4y \leq 480$
  - Profit $P = 2400x + 3400y$

45. Graph

(0,150)
(0,120)
(0,90)
(0,0)
(150,0)
(160,0)
(240,0)

Lot $x + y = 150$
Labor $3x + 4y = 480$
Capital $3x + 8y = 720$

46. Where do Capital and Labor Lines Cross?

- Write the equations:
  - Capital $3x + 8y = 720$
  - Labor $3x + 4y = 480$
  - Subtract $4y = 240$

- Divide by 4:
  - So $y = \frac{240}{4} = 60$

- Plug $y = 60$ back into the Labor equation $3x + 4y = 480$
47. Capital and Labor Line Intersection

\[ 3x + 4 \cdot 60 = 480 \]
\[ 3x + 240 = 480 \]
\[ 3x = 480 - 240 = 240 \]
\[ x = \frac{240}{3} = 80 \]

- The intersection point is (80, 60).

48. Where do Labor and Lot Lines Intersect?

- Write the equations:
  
  Labor  \[ 3x + 4y = 480 \]
  
  Lot  \[ x + y = 150 \]

- Multiply Lot equation by 3

  Capital  \[ 3x + 4y = 480 \]
  
  Lot  \[ 3x + 3y = 450 \]

- Subtract  \[ y = 30 \]

- Plug \( y = 30 \) back into the Lot equation

  \[ x + 30 = 150 \] or \( x = 120 \)

- The intersection point is (120, 30)

49. Graph

50. Solving the Problem

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>( P = 2400x + 3400y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>90</td>
<td>$306,000</td>
</tr>
<tr>
<td>80</td>
<td>60</td>
<td>$396,000</td>
</tr>
<tr>
<td>120</td>
<td>30</td>
<td>$390,000</td>
</tr>
<tr>
<td>150</td>
<td>0</td>
<td>$360,000</td>
</tr>
</tbody>
</table>

Maximum Profit

- 80 standard
- 60 deluxe
- $396,000 profit
- Note: 10 lots are unused