1. Math 210 Finite Mathematics

- Chapter 4.2
  Linear Programming Problems
  Minimization - The Dual Problem
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2. An Example

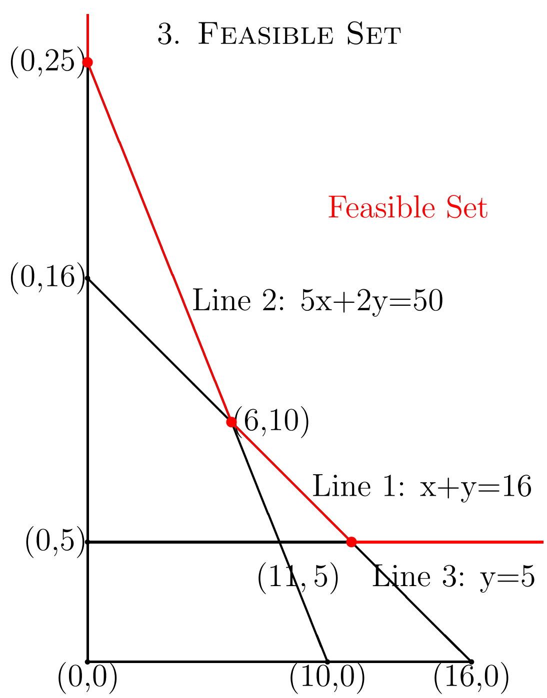
Minimize $C = 7x + 5y$ subject to

$x + y \geq 16$

$5x + 2y \geq 50$

$y \geq 5$

3. Feasible Set



4. Solving the Problem

The corner points of the feasible set are:

(0,25), (6,10), and (11,5)

x	y	$C = 7x + 5y$
0	25	125
6	10	92
11	5	102

Minimum value of C is

- 92
- and occurs at the point $(6, 10)$

Note: There is no maximum since the feasible set is infinite.

5. "Slack" variables for Minimization

When an inequality involves $\geq$, the "slack" variable should be subtracted.

For example, we rewrite the inequality $x + y \geq 16$

as $x + y - u = 16$

Really, u should perhaps be called an excess variable, rather than a slack variable,

since u represents the amount that $x + y$ exceeeds the minimum allowable requirement of 16.

Mathematically, $x + y - u = 16$ means $u = x + y - 16$

and so, $u \geq 0$ is equivalent to $x + y - 16 \geq 0$ or $x + y \geq 16$

6. Slack Variables for System

The system of inequalities

(1) $x + y \geq 16$

(2) $5x + 2y \geq 50$
(3) $y \geq 5$

can be rewritten as

(1) $x + y - u = 16$
(2) $5x + 2y - v = 50$
(3) $y - w = 5$

7. The Dual Problem

A standard minimization problem (where inequalities involve $\geq$) can be transfromed into a standard maximization problem (where inequalities involve $\leq$) by the following method:

- Write the table of data for the original problem (not the tableau with slack variables)
- Put the objective function at the bottom (without the minus signs)
- Now reverse the rows and columns to obtain a new data table
- Make the inequalites involve $\leq$
- Use u, v, ... for the "standard" variables.

8. An Example

Minimize $C = 7x + 5y$ subject to

$x + y \geq 16$

$5x + 2y \geq 50$

$y \geq 5$

Reversing the rows and columns of the data table:

x	y	const
1	1	16
5	2	50
0	1	5
7	5	C

$\Longrightarrow$

u	v	w	const
1	5	0	7
1	2	1	5
16	50	5	F

Dual: Maximize $F = 16u + 50v + 5w$ subject to

$u + 5v \leq 7$

$u + 2v + w \leq 5$

9. What's the Point?

A linear programming problem and its dual problem are related in the following important way:

> The maxiumum of the original (or primal) problem is the minimum of the dual problem, and conversely.

Moreover, the values of the variables for the solution of the dual problem can be read off the final tableau of the dual problem.

10. Example Revisited

Solving the problem

Minimize $C = 7x + 5y$ subject to

$x + y \geq 16$

$5x + 2y \geq 50$

$y \geq 5$

$x, y \geq 0$

is equivalent to solving the dual problem

Maximize $F = 16u + 50v + 5w$ subject to

$u + 5v \leq 7$

$u + 2v + w \leq 5$

$u, v, w \geq 0$

The point is: we can solve the dual problem by the Simplex Method.

11. Simplex Tableau

Maximize $F = 16u + 50v + 5w$ subject to

$u + 5v \leq 7$

$u + 2v + w \leq 5$

$u, v, w \geq 0$

Introduce the slack variables x and y

Note the change in letters!

$u + 5v + x = 7$

$u + 2v + w + y = 5$

$u, v, w, x, y \geq 0$

The Simplex Tableau:

u	v	w	x	y	F	const
1	5	0	1	0	0	7
1	2	1	0	1	0	5
-16	-50	-5	0	0	1	0

12. The Initial Tableau

u	v	w	x	y	F	const
1	5	0	1	0	0	7
1	2	1	0	1	0	5
-16	-50	-5	0	0	1	0

Since we are only using the dual to solve our original minimization problem, we are not really interested in the values of these variables in terms of the dual maximazation problem,

which we could obtain by setting the non-basic variables (u, v, and w) to zero and solving for the basic variables ($x = 7$, $y = 5$, $F = 0$).

13. The Initial Tableau

u	v	w	x	y	F	const
1	5	0	1	0	0	7
1	2	1	0	1	0	5
-16	-50	-5	0	0	1	0

We can obtain the values of the variables relating to the original minimization problem by reading off the variables on the bottom row of the tableau:

(It is not so surprising that we use the bottom row instead of the right side of the tableau, since switching rows and columns turns the original right side into the bottom.)

$u = -16 \quad v = -50 \quad w = -5 \quad x = 0 \quad y = 0$

14. Finding the pivot

The pivot column is 2 since the most negative value of the bottom row is -50 in the second column.

To find the pivot row, we compare ratios:

The Simplex Tableau:

u	v	w	x	y	F	const	ratio
1	5	0	1	0	0	7	$\frac{7}{5} = 1.2$
1	2	1	0	1	0	5	$\frac{5}{2} = 2.5$
-16	-50	-5	0	0	1	0	

The mimimum ratio of 1.2 occurs in row 1, so the pivot row is 1.

Pivot at row 1, col 2.

15. Pivot at row 1, col 2

1	5	0	1	0	0	7
1	2	1	0	1	0	5
−16	−50	−5	0	0	1	0

1	5	0	1	0	0	7
1	2	1	0	1	0	5
−16	−50	−5	0	0	1	0

Pivot at row 1 col 2

1	5	0	1	0	0	7
1	2	1	0	1	0	5
−16	−50	−5	0	0	1	0

Multiply row 1 by $\frac{1}{5}$

$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{7}{5}$
1	2	1	0	1	0	5
−16	−50	−5	0	0	1	0

$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{7}{5}$
1	2	1	0	1	0	5
-16	-50	-5	0	0	1	0

Add $-2 \times$ row 1 to row 2

$$-2 \times \frac{1}{5} + 1 = \frac{3}{5}$$

$$-2 \times 1 + 2 = 0$$

$$-2 \times \frac{1}{5} + 0 = -\frac{2}{5}$$

$$-2 \times \frac{7}{5} + 5 = \frac{11}{5}$$

$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{7}{5}$
$\frac{3}{5}$	0	1	$-\frac{2}{5}$	1	0	$\frac{11}{5}$
-16	-50	-5	0	0	1	0

$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{7}{5}$
$\frac{3}{5}$	0	1	$-\frac{2}{5}$	1	0	$\frac{11}{5}$
-16	-50	-5	0	0	1	0

Add $50 \times$ row 1 to row 3

$$50 \times \frac{1}{5} + -16 = -6$$

$$50 \times 1 + -50 = 0$$

$$50 \times \frac{1}{5} + 0 = 10$$

$$50 \times \frac{7}{5} + 0 = 70$$

$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{7}{5}$
$\frac{3}{5}$	0	1	$-\frac{2}{5}$	1	0	$\frac{11}{5}$
-6	0	-5	10	0	1	70

16. The Second Tableau

u	v	w	x	y	F	const
$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{7}{5}$
$\frac{3}{5}$	0	1	$-\frac{2}{5}$	1	0	$\frac{11}{5}$
-6	0	-5	0	0	1	70

We obtain the values of the variables relating to the original minimization problem by reading off the variables on the bottom row of the tableau:

$u = -6$

$v = 0$

$w = -5$

$x = 0$

$y = 0$

17. The Second Pivot

The pivot column is 1 since the most negative value of the bottom row is -6 in the first column.

u	v	w	x	y	F	const	ratio
$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{7}{5}$	$\frac{7}{5}/\frac{1}{5} = 7$
$\frac{3}{5}$	0	1	$-\frac{2}{5}$	1	0	$\frac{11}{5}$	$\frac{11}{5}/\frac{3}{5} = \frac{11}{3}$
-6	0	-5	10	0	1	70	

The mimimum ratio of $\frac{11}{3}$ occurs in row 2, so the pivot row is 2.

Pivot at row 2, col 1.

18. Pivot at row 2, col 1

$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{7}{5}$
$\frac{3}{5}$	0	1	$-\frac{2}{5}$	1	0	$\frac{11}{5}$
-6	0	-5	10	0	1	70

$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{7}{5}$
$\frac{3}{5}$	0	1	$-\frac{2}{5}$	1	0	$\frac{11}{5}$
-6	0	-5	10	0	1	70

Pivot at row 2 col 1

$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{7}{5}$
$\frac{3}{5}$	0	1	$-\frac{2}{5}$	1	0	$\frac{11}{5}$
-6	0	-5	10	0	1	70

Multiply row 2 by $\frac{5}{3}$

$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{7}{5}$
1	0	$\frac{5}{3}$	$-\frac{2}{3}$	$\frac{5}{3}$	0	$\frac{11}{3}$
-6	0	-5	10	0	1	70

$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{7}{5}$
1	0	$\frac{5}{3}$	$-\frac{2}{3}$	$\frac{5}{3}$	0	$\frac{11}{3}$
-6	0	-5	10	0	1	70

Add $-\frac{1}{5}$ $\times$ row 2 to row 1

$$-\frac{1}{5} \times 1 + \frac{1}{5} = 0$$

$$-\frac{1}{5} \times \frac{5}{3} + 0 = -\frac{1}{3}$$

$$-\frac{1}{5} \times -\frac{2}{3} + \frac{1}{5} = \frac{1}{3}$$

$$-\frac{1}{5} \times \frac{5}{3} + 0 = -\frac{1}{3}$$

$$-\frac{1}{5} \times \frac{11}{3} + \frac{7}{5} = \frac{2}{3}$$

0	1	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{2}{3}$
1	0	$\frac{5}{3}$	$-\frac{2}{3}$	$\frac{5}{3}$	0	$\frac{11}{3}$
-6	0	-5	10	0	1	70

0	1	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{2}{3}$
1	0	$\frac{5}{3}$	$-\frac{2}{3}$	$\frac{5}{3}$	0	$\frac{11}{3}$
-6	0	-5	10	0	1	70

Add 6 $\times$ row 2 to row 3

$$6 \times 1 + -6 = 0$$

$$6 \times \frac{5}{3} + -5 = 5$$

$$6 \times -\frac{2}{3} + 10 = 6$$

$$6 \times \frac{5}{3} + 0 = 10$$

$$6 \times \frac{11}{3} + 70 = 92$$

0	1	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{2}{3}$
1	0	$\frac{5}{3}$	$-\frac{2}{3}$	$\frac{5}{3}$	0	$\frac{11}{3}$
0	0	5	6	10	1	92

19. The Third Tableau - Dual Look

u	v	w	x	y	F	const
0	1	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{2}{3}$
1	0	$\frac{5}{3}$	$-\frac{2}{3}$	$\frac{5}{3}$	0	$\frac{11}{3}$
0	0	5	6	10	1	92

Look at the colored values at the bottom of the tableau.

These give the values of x, y, and P which solve the original minimization problem:

$x = 6 \quad y = 10 \quad F = 92$

Since the maximum of the dual equals the minimum of the original problem, the minimum value of C in the original problem is: $\min C = \max F = 92$

The slack variables are: $u = 0 \quad v = 0 \quad w = 5$

Pure Magic

20. The Nutrition Problem

Minimize $C = 21x + 14y$ subject to

$2x + 3y \geq 12$

$3x + y \geq 6$

$x + 3y \geq 9$

$x, y \geq 0$

To obtain the dual problem, we reverse the rows and columns of the data table.

21. Dual of the Nutrition Problem

Minimize $C = 21x + 14y$ subject to

$2x + 3y \geq 12$

$3x + y \geq 6$

$x + 3y \geq 9$

x	y	const
2	3	12
3	1	6
1	3	9
21	14	C

$\Longrightarrow$

u	v	w	const
2	3	1	21
3	1	3	14
12	6	9	F

Dual: Maximize $F = 12u + 6v + 9w$ subject to

$2u + 3v + w \leq 21$

$3 + v + 3w \leq 14$

22. Simplex Tableau

Maximize $F = 12u + 6v + 9w$ subject to

$2u + 3v + w \leq 21$

$3u + v + 3w \leq 14$

$u, v, w \geq 0$

Introduce the slack variables x and y

Note the change in letters!

$2u + 3v + w + x = 21$

$3u + v + 3w + y = 14$

$u, v, w, x, y \geq 0$

The Simplex Tableau:

u	v	w	x	y	F	const
2	3	1	1	0	0	21
3	1	3	0	1	0	14
-12	-6	-9	0	0	1	0

23. Finding the pivot

The pivot column is 1 since the most negative value of the bottom row is -12 in the first column.

To find the pivot row, we compare ratios:

The Simplex Tableau:

u	v	w	x	y	F	const	ratio
2	3	1	1	0	0	21	$\frac{21}{2} = 10.5$
3	1	3	0	1	0	14	$\frac{14}{3} = 4\frac{2}{3}$
-12	-6	-9	0	0	1	0	

The mimimum ratio of $4\frac{2}{3}$ occurs in row 2, so the pivot row is 2.

Pivot at row 2, col 1.

24. Pivot at row 2, col 1

2	3	1	1	0	0	21
3	1	3	0	1	0	14
−12	−6	−9	0	0	1	0

2	3	1	1	0	0	21
3	1	3	0	1	0	14
−12	−6	−9	0	0	1	0

Pivot at row 2 col 1

2	3	1	1	0	0	21
3	1	3	0	1	0	14
−12	−6	−9	0	0	1	0

Multiply row 2 by $\frac{1}{3}$

2	3	1	1	0	0	21
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$
-12	-6	-9	0	0	1	0

2	3	1	1	0	0	21
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$
-12	-6	-9	0	0	1	0

Add $-2 \times$ row 2 to row 1

$$-2 \times 1 + 2 = 0$$

$$-2 \times \frac{1}{3} + 3 = \frac{7}{3}$$

$$-2 \times 1 + 1 = -1$$

$$-2 \times \frac{1}{3} + 0 = -\frac{2}{3}$$

$$-2 \times \frac{14}{3} + 21 = \frac{35}{3}$$

0	$\frac{7}{3}$	-1	1	$-\frac{2}{3}$	0	$\frac{35}{3}$
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$
-12	-6	-9	0	0	1	0

0	$\frac{7}{3}$	-1	1	$-\frac{2}{3}$	0	$\frac{35}{3}$
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$
-12	-6	-9	0	0	1	0

Add $12 \times$ row 2 to row 3

$$12 \times 1 + -12 = 0$$

$$12 \times \frac{1}{3} + -6 = -2$$

$$12 \times 1 + -9 = 3$$

$$12 \times \frac{1}{3} + 0 = 4$$

$$12 \times \frac{14}{3} + 0 = 56$$

0	$\frac{7}{3}$	-1	1	$-\frac{2}{3}$	0	$\frac{35}{3}$
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$
0	-2	3	0	4	1	56

25. The Second Tableau

u	v	w	x	y	F	const
0	$\frac{7}{3}$	-1	1	$-\frac{2}{3}$	0	$\frac{35}{3}$
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$
0	-2	3	0	4	1	56

We obtain the values of the variables relating to the original minimization problem by reading off the variables on the bottom row of the tableau:

$u = 0$

$v = -2$

$w = 3$

$x = 0$

$y = 4$

26. Finding the second pivot

The pivot column is 2 since the most negative value of the bottom row is -2 in the first column.

To find the pivot row, we compare ratios:

The Simplex Tableau:

u	v	w	x	y	F	const	ratio
0	$\frac{7}{3}$	-1	1	$-\frac{2}{3}$	0	$\frac{35}{3}$	$\frac{35}{3}/\frac{7}{3} = 5$
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$	$\frac{14}{3}/\frac{1}{3} = 14$
0	-2	3	0	4	1	56	

The mimimum ratio of 5 occurs in row 1, so the pivot row is 1.

Pivot at row 1, col 2.

27. Pivot at row 1, col 2

0	$\frac{7}{3}$	-1	1	$-\frac{2}{3}$	0	$\frac{35}{3}$
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$
0	-2	3	0	4	1	56

0	$\frac{7}{3}$	-1	1	$-\frac{2}{3}$	0	$\frac{35}{3}$
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$
0	-2	3	0	4	1	56

Pivot at row 1 col 2

0	$\frac{7}{3}$	-1	1	$-\frac{2}{3}$	0	$\frac{35}{3}$
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$
0	-2	3	0	4	1	56

Multiply row 1 by $\frac{3}{7}$

0	1	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	0	5
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$
0	-2	3	0	4	1	56

0	1	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	0	5
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$
0	-2	3	0	4	1	56

Add $-\frac{1}{3} \times$ row 1 to row 2

$$-\frac{1}{3} \times 1 + \frac{1}{3} = 0$$

$$-\frac{1}{3} \times -\frac{3}{7} + 1 = \frac{8}{7}$$

$$-\frac{1}{3} \times \frac{3}{7} + 0 = -\frac{1}{7}$$

$$-\frac{1}{3} \times -\frac{2}{7} + \frac{1}{3} = \frac{3}{7}$$

$$-\frac{1}{3} \times 5 + \frac{14}{3} = 3$$

0	1	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	0	5
1	0	$\frac{8}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$	0	3
0	-2	3	0	4	1	56

0	1	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	0	5
1	0	$\frac{8}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$	0	3
0	-2	3	0	4	1	56

Add 2 × row 1 to row 3

$2 \times 1 + -2 = 0$

$2 \times -\frac{3}{7} + 3 = \frac{15}{7}$

$2 \times \frac{3}{7} + 0 = \frac{6}{7}$

$2 \times -\frac{2}{7} + 4 = \frac{24}{7}$

$2 \times 5 + 56 = 66$

0	1	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	0	5
1	0	$\frac{8}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$	0	3
0	0	$\frac{15}{7}$	$\frac{6}{7}$	$\frac{24}{7}$	1	66

28. The Third Tableau

u	v	w	x	y	F	const
0	1	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	0	5
1	0	$\frac{8}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$	0	3
0	0	$\frac{15}{7}$	$\frac{6}{7}$	$\frac{24}{7}$	1	66

The values of x, y, and C which solve the original minimization problem can be read from the bottom row:

$x = \frac{6}{7}$

$y = \frac{24}{7}$

Since this is the final tableau, the minimum value of C of the original minimization problem is the maximum value of F for the dual problem:

$C = 66$

29. Feasible Region