1. Math 210 Finite Mathematics

- Chapter 4.3
  Linear Programming Problems
  Non Standard Problems
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2. The Transportation Problem

Minimize the cost $C = -3x - 2y + 375$

subject to the constraints

- $x \leq 25$, $y \leq 30$
- $15 \leq x + y$
- $x + y \leq 45$
- $x \geq 0$, $y \geq 0$

This is an example of a non-standard problem. Some inequalities are $\leq$ and some are $\geq$.

3. Two Tricks

To convert a problem in non-standard form into standard form we utulize two tricks:

- Minimizing F is equivalent to maximizing $-F$
- and vice-versa
- You can change an inequality from $\geq$ to $\leq$ by multiplying both sides by -1
- A similarly trick changes from $\leq$ to $\geq$

4. Applying the Two Tricks

To convert the transportation problem into standard form we utulize these tricks:

Instead of minimizing $C = -3x - 2y + 375$,

maximize the negative $F = -C = 3x + 2y - 375$

All but one constraining inequality has the correct direction for maximization ($\leq$): $x + y \geq 15$

Multiplying this inequality by -1 gives

$-x + -y \leq -15$

5. Revised Transportation Problem

Maximize the function $F = 3x + 2y - 375$

subject to the constraints

- $x \leq 25$, $y \leq 30$
- $x + y \leq 45$
- $-x + -y \leq -15$
- $x \geq 0$, $y \geq 0$

Using slack variables u, v, w, and t:

- $x + u = 25$, $y + v = 30$
- $x + y + w = 45$
- $-x + -y + t = -15$
- $x \geq 0$, $y \geq 0$

6. Simplex Tableau

x	y	u	v	w	t	F	const
1	0	1	0	0	0	0	25
0	1	0	1	0	0	0	30
1	1	0	0	1	0	0	45
−1	−1	0	0	0	1	0	−15
−3	−2	0	0	0	0	1	−375

Setting the non-basic variables (x and y) to zero gives the following values for the basic variables:

$u = 25 \quad v = 30 \quad w = 45 \quad t = -15$

OOPS! One of our slack variables is negative. This means that the initial point $(x, y) = (0, 0)$ is not in the feasible set.

7. Fixing a non-Standard Tableau

When an entry in the constant row (not at the bottom) is negative, we can fix it by the following procedure:

- Select one of the negative entries in its row. The column containing this entry will be the pivot column.
- Select the pivot row by computing the positive ratio of elements in the constant column and elements in the pivot column (except at the bottom). The smallest positive ratio determines the pivot row.
- Pivot
- Repeat until no negative entries are found in the constant column of the resulting tableau.

8. Finding Pivot Column

x	y	u	v	w	t	F	const
1	0	1	0	0	0	0	25
0	1	0	1	0	0	0	30
1	1	0	0	1	0	0	45
−1	−1	0	0	0	1	0	−15
−3	−2	0	0	0	0	1	−375

Since −15 is the only negative entry in the constant column (We do not count −375 at the bottom)

we scan this row looking for another negative entry.

We find two: in columns 1 and column 2.

Flipping a coin, we choose column 1 to be the pivot column.

9. Finding the Pivot Row

x	y	u	v	w	t	F	const	ratio
1	0	1	0	0	0	0	25	$\frac{25}{1} = 25$
0	1	0	1	0	0	0	30	none
1	1	0	0	1	0	0	45	$\frac{45}{1} = 45$
−1	−1	0	0	0	1	0	−15	$\frac{-15}{-1} = 15$
−3	−2	0	0	0	0	1	−375	

Since the smallest positive ratio 15 occurs in row 4, the pivot row is 4.

Pivot at row 4, col 1.

10. Pivot at row 4, col 1

1	0	1	0	0	0	0	25
0	1	0	1	0	0	0	30
1	1	0	0	1	0	0	45
−1	−1	0	0	0	1	0	−15
−3	−2	0	0	0	0	1	−375

1	0	1	0	0	0	0	25
0	1	0	1	0	0	0	30
1	1	0	0	1	0	0	45
-1	-1	0	0	0	1	0	-15
-3	-2	0	0	0	0	1	-375

Pivot at row 4 col 1

1	0	1	0	0	0	0	25
0	1	0	1	0	0	0	30
1	1	0	0	1	0	0	45
-1	-1	0	0	0	1	0	-15
-3	-2	0	0	0	0	1	-375

Multiply row 4 by -1

1	0	1	0	0	0	0	25
0	1	0	1	0	0	0	30
1	1	0	0	1	0	0	45
1	1	0	0	0	−1	0	15
−3	−2	0	0	0	0	1	−375

1	0	1	0	0	0	0	25
0	1	0	1	0	0	0	30
1	1	0	0	1	0	0	45
1	1	0	0	0	−1	0	15
−3	−2	0	0	0	0	1	−375

Add $-1 \times$ row 4 to row 1

$$-1 \times 1 + 1 = 0$$

$$-1 \times 1 + 0 = -1$$

$$-1 \times -1 + 0 = 1$$

$$-1 \times 15 + 25 = 10$$

0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
1	1	0	0	1	0	0	45
1	1	0	0	0	-1	0	15
-3	-2	0	0	0	0	1	-375

0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
1	1	0	0	1	0	0	45
1	1	0	0	0	-1	0	15
-3	-2	0	0	0	0	1	-375

Add $0 \times$ row 4 to row 2

$$0 \times 1 + 0 = 0$$

$$0 \times 1 + 1 = 1$$

$$0 \times -1 + 0 = 0$$

$$0 \times 15 + 30 = 30$$

0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
1	1	0	0	1	0	0	45
1	1	0	0	0	-1	0	15
-3	-2	0	0	0	0	1	-375

0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
1	1	0	0	1	0	0	45
1	1	0	0	0	-1	0	15
-3	-2	0	0	0	0	1	-375

Add $-1 \times$ row 4 to row 3

$$-1 \times 1 + 1 = 0$$

$$-1 \times 1 + 1 = 0$$

$$-1 \times -1 + 0 = 1$$

$$-1 \times 15 + 45 = 30$$

0	−1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	0	0	0	1	1	0	30
1	1	0	0	0	−1	0	15
−3	−2	0	0	0	0	1	−375

0	−1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	0	0	0	1	1	0	30
1	1	0	0	0	−1	0	15
−3	−2	0	0	0	0	1	−375

Add 3 × row 4 to row 5

$3 \times 1 + -3 = 0$

$3 \times 1 + -2 = 1$

$3 \times -1 + 0 = -3$

$3 \times 15 + -375 = -330$

0	−1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	0	0	0	1	1	0	30
1	1	0	0	0	−1	0	15
0	1	0	0	0	−3	1	−330

11. Second Simplex Tableau

x	y	u	v	w	t	F	const
0	−1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	0	0	0	1	1	0	30
1	1	0	0	0	−1	0	15
0	1	0	0	0	−3	1	−330

Since the only negative entry in the bottom row (we don't count -330) is the -3 in column 6, the pivot column is 6.

12. Finding Pivot Row

x	y	u	v	w	t	F	const	ratio
0	−1	1	0	0	1	0	10	$\frac{10}{1} = 10$
0	1	0	1	0	0	0	30	none
0	0	0	0	1	1	0	30	$\frac{30}{1} = 30$
1	1	0	0	0	−1	0	15	none
0	1	0	0	0	−3	1	−330	

Since the minimum ratio of 10 occurs in row 1, the pivot row is 1.

Pivot at row 1, column 6.

13. Pivot at row 1, col 6

0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	0	0	0	1	1	0	30
1	1	0	0	0	-1	0	15
0	1	0	0	0	-3	1	-330

0	-1	1	0	0	$\boxed{1}$	0	10
0	1	0	1	0	0	0	30
0	0	0	0	1	1	0	30
1	1	0	0	0	-1	0	15
0	1	0	0	0	-3	1	-330

Pivot at row 1 col 6

0	−1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	0	0	0	1	1	0	30
1	1	0	0	0	−1	0	15
0	1	0	0	0	−3	1	−330

Add 0 × row 1 to row 2

$$0 \times -1 + 1 = 1$$

$$0 \times 1 + 0 = 0$$

$$0 \times 1 + 0 = 0$$

$$0 \times 10 + 30 = 30$$

0	−1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	0	0	0	1	1	0	30
1	1	0	0	0	−1	0	15
0	1	0	0	0	−3	1	−330

0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	0	0	0	1	1	0	30
1	1	0	0	0	-1	0	15
0	1	0	0	0	-3	1	-330

Add $-1 \times$ row 1 to row 3

$-1 \times -1 + 0 = 1$

$-1 \times 1 + 0 = -1$

$-1 \times 1 + 1 = 0$

$-1 \times 10 + 30 = 20$

0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	1	-1	0	1	0	0	20
1	1	0	0	0	-1	0	15
0	1	0	0	0	-3	1	-330

0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	1	-1	0	1	0	0	20
1	1	0	0	0	-1	0	15
0	1	0	0	0	-3	1	-330

Add $1 \times$ row 1 to row 4

$$1 \times -1 + 1 = 0$$

$$1 \times 1 + 0 = 1$$

$$1 \times 1 + -1 = 0$$

$$1 \times 10 + 15 = 25$$

0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	1	-1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	1	0	0	0	-3	1	-330

0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	1	-1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	1	0	0	0	-3	1	-330

Add 3 × row 1 to row 5

$3 \times -1 + 1 = -2$

$3 \times 1 + 0 = 3$

$3 \times 1 + -3 = 0$

$3 \times 10 + -330 = -300$

0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	1	-1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	-2	3	0	0	0	1	-300

14. Third Simplex Tableau

x	y	u	v	w	t	F	const
0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	1	-1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	-2	3	0	0	0	1	-300

Setting the non-basic variables y and u to zero, the values of the basic variables are:

$x = 25,\ v = 30,\ w = 20,\ t = 10,\ F = -300$

Since the only negative entry in the bottom row (to the left of -330) is the -2 in column 2, the pivot column is 2.

15. Find pivot row

x	y	u	v	w	t	F	const	ratio
0	-1	1	0	0	1	0	10	none
0	1	0	1	0	0	0	30	$\frac{30}{1} = 30$
0	1	-1	0	1	0	0	20	$\frac{20}{1} = 20$
1	0	1	0	0	0	0	25	none
0	-2	3	0	0	0	1	-300	

Since the minimum ratio is 20 occurs in row 3, the pivot row is 3.

Pivot at row 3, column 2.

16. Pivot at row 3, col 2

0	−1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	1	−1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	−2	3	0	0	0	1	−300

0	−1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	1	−1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	−2	3	0	0	0	1	−300

Pivot at row 3 col 2

0	−1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	1	−1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	−2	3	0	0	0	1	−300

Add 1 × row 3 to row 1

$$1 \times 1 + -1 = 0$$

$$1 \times -1 + 1 = 0$$

$$1 \times 1 + 0 = 1$$

$$1 \times 20 + 10 = 30$$

0	0	0	0	1	1	0	30
0	1	0	1	0	0	0	30
0	1	−1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	−2	3	0	0	0	1	−300

0	0	0	0	1	1	0	30
0	1	0	1	0	0	0	30
0	1	−1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	−2	3	0	0	0	1	−300

Add $-1 \times$ row 3 to row 2

$$-1 \times 1 + 1 = 0$$

$$-1 \times -1 + 0 = 1$$

$$-1 \times 1 + 0 = -1$$

$$-1 \times 20 + 30 = 10$$

0	0	0	0	1	1	0	30
0	0	1	1	−1	0	0	10
0	1	−1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	−2	3	0	0	0	1	−300

0	0	0	0	1	1	0	30
0	0	1	1	−1	0	0	10
0	1	−1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	−2	3	0	0	0	1	−300

Add $0 \times$ row 3 to row 4

$$0 \times 1 + 0 = 0$$

$$0 \times -1 + 1 = 1$$

$$0 \times 1 + 0 = 0$$

$$0 \times 20 + 25 = 25$$

0	0	0	0	1	1	0	30
0	0	1	1	−1	0	0	10
0	1	−1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	−2	3	0	0	0	1	−300

0	0	0	0	1	1	0	30
0	0	1	1	−1	0	0	10
0	1	−1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	−2	3	0	0	0	1	−300

Add 2 × row 3 to row 5

$2 \times 1 + -2 = 0$

$2 \times -1 + 3 = 1$

$2 \times 1 + 0 = 2$

$2 \times 20 + -300 = -260$

0	0	0	0	1	1	0	30
0	0	1	1	−1	0	0	10
0	1	−1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	0	1	0	2	0	1	−260

17. Final Simplex Tableau

x	y	u	v	w	t	F	const
0	0	0	0	1	1	0	30
0	0	1	1	-1	0	0	10
0	1	-1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	0	1	0	2	0	1	-260

Setting the non-basic variables u and w to zero, the values of the basic variables are:

$x = 25$, $y = 20$, $v = 10$, $t = 30$, $F = -260$

Since there are no negative entries in the bottom row (before the constant column), we are done.

Since the cost function C of the original transportation problem is the negative of F,

Minimum cost is \$260 at the point $(x, y) = (25, 20)$

18. Three Types of Problems

(1) Standard Form Maximization
- Objective Function is maximized
- All inequalities are $\leq$ (except $x, y \geq 0$)
- Use Standard Simplex Method
- Prototype: Construction Problem

(2) Standard Form Minimization
- Objective Function is minimized
- All inequalities are $\geq$
- Reverse rows and columns of the data table to obtain the dual Problem
- Use Simplex Method on Dual Problem
- Prototype: Nutrition Problem

(3) Non-Standard Form
- Inequalities are mixed (some $\leq$, some $\geq$)
- Convert the problem to standard form
- Pivot to remove negative entries in the constant column
- Prototype: Transportation Problem

19. Standard Form Problem

Construction Problem

Maximize profit $P = 2400x + 3400y$ subject to

$x + y \leq 150$

$3x + 8y \leq 720$

$3x + 4y \leq 480$

$x, y \geq 0$

This problem is in standard form: it is a maximization problem and all inequalities are $\leq$.

20. Construction Tableau #1

x	y	u	v	w	P	const
1	1	1	0	0	0	150
3	8	0	1	0	0	720
3	4	0	0	1	0	480
-2400	-3400	0	0	0	1	0

$x = 0 \quad y = 0 \quad P = 0$

Pivot at row 2, col 2.

21. Construction Tableau #2

x	y	u	v	w	P	const
$\frac{5}{8}$	0	1	$-\frac{1}{8}$	0	0	60
$\frac{3}{8}$	1	0	$\frac{1}{8}$	0	0	90
$\frac{3}{2}$	0	0	$-\frac{1}{2}$	1	0	120
-1125	0	0	425	0	1	$306K$

$x = 0 \quad y = 90 \quad P = 306\text{K}$

Pivot at row 3, col 1.

22. Construction Tableau #3

x	y	u	v	w	P	const
0	0	1	$\frac{1}{12}$	$-\frac{5}{12}$	0	10
0	1	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	60
1	0	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	80
0	0	0	50	750	1	$396K$

$$x = 80 \quad y = 60$$

$$P = 396,000 \text{ (maximum)}$$

23. Dual Problem

The Nutrition Problem

Minimize $C = 21x + 14y$ subject to

$2x + 3y \geq 12$

$3x + y \geq 6$

$x + 3y \geq 9$

$x, y \geq 0$

Dual:

Maximize $F = 12u + 6v + 9w$ subject to

$2u + 3v + w \leq 21$

$3 + v + 3w \leq 14$

$u, v, w \geq 0$

24. Nutrition Tableau #1

u	v	w	x	y	F	const
2	3	1	1	0	0	21
3	1	3	0	1	0	14
-12	-6	-9	0	0	1	0

Value of the variables for the original minimization are read from the bottom of the tableau:

$$x = 0 \quad y = 0$$

25. Nutrition Tableau #2

u	v	w	x	y	F	const
0	$\frac{7}{3}$	-1	1	$-\frac{2}{3}$	0	$\frac{35}{3}$
1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$
0	-2	3	10	4	1	56

$$x = 10 \quad y = 4$$

26. Nutrition Tableau #3

u	v	w	x	y	F	const
0	1	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	0	5
1	0	$\frac{8}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$	0	3
0	0	$\frac{15}{7}$	$\frac{6}{7}$	$\frac{24}{7}$	1	66

$$x = \frac{6}{7}, \quad y = \frac{24}{7}$$

By principal of duality,

minimum C = maximum $F = 66$

27. Non Standard Problem

Original Transportation Problem:

Minimize the cost $C = -3x - 2y + 375$ subject to

$x \leq 25$, $y \leq 30$

$x + y \geq 15$

$x + y \leq 45$

$x \geq 0$, $y \geq 0$

Non-standard because some inequalities are $\leq$ and some are $\geq$.

Revised Transportation Problem:

Maximize the function $F = 3x + 2y - 375$ subject to

$x \leq 25$, $y \leq 30$

$-x + -y \leq -15$

$x + y \leq 45$

$x \geq 0$, $y \geq 0$

28. TRANSPORTATION TABLEAU #1

x	y	u	v	w	t	F	const
1	0	1	0	0	0	0	25
0	1	0	1	0	0	0	30
1	1	0	0	1	0	0	45
-1	-1	0	0	0	1	0	-15
-3	-2	0	0	0	0	1	-375

$x = 0 \quad y = 0 \quad F = -375$

Conflict: -15 in constant column

Resolve conflict by pivoting at row 4, col 1.

29. TRANSPORTATION TABLEAU #2

x	y	u	v	w	t	F	const
0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	0	0	0	1	1	0	30
1	1	0	0	0	-1	0	15
0	1	0	0	0	-3	1	-330

$x = 15 \quad y = 0 \quad F = -330$
no negatives in constant column
(above bottom row)
Pivot at row 1, col 6.

30. Transportation Tableau #3

x	y	u	v	w	t	F	const
0	-1	1	0	0	1	0	10
0	1	0	1	0	0	0	30
0	1	-1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	-2	3	0	0	0	1	-300

$x = 25 \quad y = 0 \quad F = -300$

Pivot at row 3, col 2.

31. Transportation Tableau #4

x	y	u	v	w	t	F	const
0	0	0	0	1	1	0	30
0	0	1	1	-1	0	0	10
0	1	-1	0	1	0	0	20
1	0	1	0	0	0	0	25
0	0	1	0	2	0	1	-260

$x = 25 \quad y = 20 \quad F = -260$

Solution: Since $C = -F$,

Minimum cost is \$260