

MATH 210 LECTURE NOTES:
CHAPTER 2.3 MATRICES

Richard Blecksmith
Dept. of Mathematical Sciences
Northern Illinois University

1. ROW REDUCED FORM

A matrix is in **row-reduced form** if

- (1) All zero rows lie at the bottom of the matrix.
- (2) The first entry of a nonzero row is a pivot.
 - (a) The value of each pivot is 1.
 - (b) All other entries in the pivot column are 0.
- (3) The first nonzero entry of each row is to the right of the first nonzero entry of the previous row.

2. ROW REDUCED FORM ?

Matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Not in reduced form.

The entry in row 2, col 3 is not a pivot, because there is a non-zero entry (2) in the third column.

3. ROW REDUCED FORM ?

Matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Yes, in reduced form.

4. ROW REDUCED FORM ?

Matrix

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

No, not reduced form.

Leading non-zero entry in row 2 is to the left of the leading non-zero entry in row 1.

5. ROW REDUCED FORM ?

Matrix

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

No, not reduced.

Leading non-zero entry of row 2 is not 1.

6. UNIQUENESS

Systematically applying the three row reduction operations to a given matrix A

- (1) switch rows
- (2) multiply a row by a nonzero constant
- (3) add a multiple of one row to another

allows us to transform A into a **row reduced** matrix A' .

It turns out: A' is unique.

That is, if two people reduce the matrix A , the intermediary steps may be quite different, but (barring mistakes) the final reduced matrices will be the same.

7. FACTORING ANALOGY

Factoring a number into primes is like this. The intermediate steps may be quite different.

For example, there are three different ways to begin factoring the number 30:

1. $30 = 2 \times 15 = 2 \times (3 \times 5)$
2. $30 = 3 \times 10 = 3 \times (2 \times 5)$
3. $30 = 5 \times 6 = 5 \times (2 \times 3)$

Factoring 15 in the first equation gives

Factoring 10 in the second equation gives

Factoring 6 in the third equation gives

We always end up with the same set of primes 2, 3, 5 (possibly in a different order).

That is, the set of primes obtained from factoring a number is unique.

8. THE THREE POSSIBILITIES

When a matrix, representing a system of equations, is row reduced to matrix A' , there are three possible conclusions:

- (1) the system is **inconsistent**, that is, the equations have no solution
- (2) the system has a **unique** solution, which can easily be read off from the final column of A' .
- (3) the system has **infinitely many** solutions.

9. NO SOLUTION

If one row of A' looks like

$$0 \ 0 \ 0 \ \cdots \ 0 \ 0 \ d$$

where the constant d is non-zero, then the system is **inconsistent**, that is, no solution exists.

This row corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 + \cdots + 0x_n = d$$

or $0 = d$, which is, of course, completely false unless d is zero.

10. CONSISTENT SOLUTIONS

If you cannot find an inconsistent row in the reduced matrix A' , that is a row consisting of a string of 0's followed by a non-zero constant, then the system is consistent.

There are solutions. The question is: How many?

Count the number of non-zero rows of the reduced matrix A' .

This number will always be less than or equal to the number of variables of the original system.

If the number of non-zero rows of A' equals the number of variables in the system, then the system has a **unique** solution, which can easily be read off from the final column of A' .

Otherwise, the system has a **infinitely many** solutions.

11. THE THREE EXAMPLES

$$\begin{bmatrix} 3 & -1 & -1 & 1 \\ 7 & 1 & -1 & 6 \\ 2 & 1 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{10}{3} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \quad \begin{array}{l} \text{Unique} \\ \text{Solution} \end{array}$$

$$\begin{bmatrix} 3 & -1 & 1 & 1 \\ 7 & 1 & -1 & 6 \\ 2 & 1 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{7}{10} \\ 0 & 1 & -1 & \frac{11}{10} \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad \begin{array}{l} \text{No} \\ \text{Solution} \end{array}$$

$$\begin{bmatrix} 3 & -1 & -1 & 1 \\ 7 & 1 & -1 & 5 \\ 2 & 1 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{3}{5} \\ 0 & 1 & -1 & \frac{4}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \text{Infinite} \\ \text{Solutions} \end{array}$$

12. UNIQUE SOLUTION

You cannot tell beforehand that a system of equations has a unique solution.

For example, suppose you have three equations in the x - y plane. You may expect that the three lines will cross at three different points, forming a triangle.

In this case the system would be inconsistent, since the three lines do not all pass through a common point.

But you cannot be sure.

Putting the equations into a matrix will resolve the question.

13. EXAMPLE

Consider the system of equations:

$$3x - 2y = 5$$

$$-x + 3y = -4$$

$$2x - 4y = 6$$

The matrix of the system is

$$A = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 3 & -4 \\ 2 & -4 & 6 \end{bmatrix}$$

Switch rows 1 and 2:

$$\begin{bmatrix} -1 & 3 & -4 \\ 3 & -2 & 5 \\ 2 & -4 & 6 \end{bmatrix}$$

Multiply row 1 by -1 :

14. EXAMPLE CONT'D

$$\begin{bmatrix} 1 & -3 & 4 \\ \boxed{3} & -2 & 5 \\ 2 & -4 & 6 \end{bmatrix}$$

Add $-3 \times$ row 1 to row 2:

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 7 & -7 \\ \boxed{2} & -4 & 6 \end{bmatrix}$$

Add $-2 \times$ row 1 to row 3:

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 7 & -7 \\ 0 & 2 & -2 \end{bmatrix}$$

Multiply row 2 by $\frac{1}{7}$:

15. EXAMPLE CONT'D

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -1 \\ 0 & \boxed{2} & -2 \end{bmatrix}$$

Add $-2 \times$ row 2 to row 3:

$$\begin{bmatrix} 1 & \boxed{-3} & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Finally, add $3 \times$ row 2 to row 1:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

16. EXAMPLE CONCLUSION

$$A = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 3 & -4 \\ 2 & -4 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

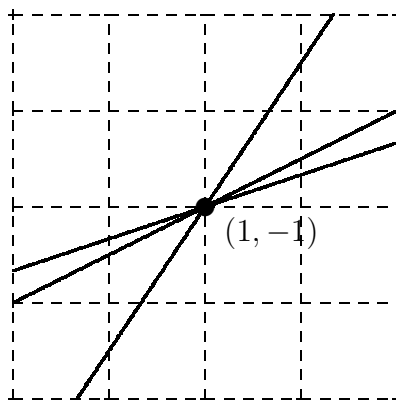
The number of nonzero rows in the reduced matrix is 2, which equals the number of variables in the system.

Conclusion: Unique Solution: $x = 1, y = -1$

We can check our answer by plugging these values into the original equation

$$\begin{array}{ll} 3x - 2y = 5 & 3 \cdot 1 - 2 \cdot (-1) = 5 \\ -x + 3y = -4 & -1 + 3 \cdot (-1) = -4 \\ 2x - 4y = 6 & 2 \cdot 1 - 4 \cdot (-1) = 6 \end{array}$$

17. GEOMETRIC EXPLANATION



Line 1: $3x - 2y = 5$

Line 2: $-x + 3y = -4$

Line 3: $2x - 4y = 6$

Suppose a matrix A row reduces to

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 5 \\ 0 & 0 & 1 & -4 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What are the solutions to the original system of equations?

The system is consistent since there are no rows of the form $[0 \ 0 \ \cdots \ 0 \ d]$, with $d \neq 0$.

Since the number of nonzero rows of the reduced matrix is 2 and the number of variables is 4 (one less than the number of columns of the matrix), there are infinitely many solutions.

How can we describe them?

19. INFINITE EXAMPLE CONT'D

The two equations corresponding to the two rows of the reduced matrix are:

$$x_1 - 2x_2 + 3x_4 = 5$$

$$x_3 - 4x_4 = 6$$

These can be rewritten as

$$x_1 = 2x_2 - 3x_4 + 5$$

$$x_3 = 4x_4 + 6$$

The variables x_1 and x_3 are called **dependent** variables since they **depend** on the values assigned to x_2 and x_4 , called the **independent** variables. The dependent variables correspond to the pivot columns of the reduced matrix.

20. INFINITE EXAMPLE CONT'D

We express the fact that the independent variables can be whatever values you please by adding the equations

$$x_2 = x_2$$

$$x_4 = x_4$$

to our system:

$$x_1 = 2x_2 - 3x_4 + 5$$

$$x_2 = x_2$$

$$x_3 = 4x_4 + 6$$

$$x_4 = x_4$$

We often replace the variables x_2 and x_4 with letters such as s and t :

21. INFINITE EXAMPLE FINALE

$$x_1 = 2s - 3t + 5$$

$$x_2 = s$$

$$x_3 = 4t + 6$$

$$x_4 = t$$

We get infinitely many solutions by plugging in values for the independent variables s and t .

22. DEFINITION OF MATRIX

It is often convenient to arrange information in table form.

A rectangular table of numbers is called a **matrix**.

The dimensions m by n of the table signify that

m = the number of rows of the matrix

n = the number of columns of the matrix.

23. SPREADSHEETS

The most familiar type of matrix is a spreadsheet.

All the grades for this class are stored in a matrix (or spreadsheet).

Each row of the matrix represents one student in the class.

Each column represents a Test Score, Recitation Grade, or Final Exam Score.

24. BASEBALL BOX SCORE

The box score of a baseball game is a 2×9 matrix. The columns represent the number of runs scored in that inning. Visitors runs are listed in the first row; the home team by the second row.

For example in the Sox play the Cubs at Wrigley:

Sox	1	0	2	0	0	1	1	0	2
Cubs	0	0	0	3	0	0	1	0	2

Who won?

25. CONSTRUCTION PROBLEM

A contractor builds two types of homes: the standard model and the deluxe model. The standard model requires one lot, \$12,000 capital, 150 labor-days to build, and is sold for a profit of \$2400. The deluxe model requires one lot, \$32,000 capital, 200 labor-days to build, and is sold for a profit of \$3400. The contractor has 150 lots. The bank is willing to loan him \$2,880,000 for the project and he has a maximum labor force available of 24,000 labor-days.

How many houses should he build to realize the greatest profit?

26. ORGANIZING THE DATA

Resource	Standard	Deluxe	Available
Lot	1	1	150
Capital	12,000	32,000	2,880,000
Labor	150	200	24,000
Profit	2400	3400	P

27. ALGEBRA OF MATRICES

Given two matrices A and B , we can

- compare them for equality
- add them
- subtract them

- multiply them
- multiply a matrix by a number

28. ADDING MATRICES

Dimension Check: Are the dimensions of A and B identical?

If not, stop, you cannot add them.

If $\dim(A) = \dim(B) = m$ by n , then the dimension of $A + B$ is also m by n .

To add the matrices, simply add the corresponding entries.

29. ADDITION EXAMPLE

$$\begin{bmatrix} 2 & 8 & -4 & 6 \\ -7 & 0 & -14 & 5 \\ 12 & -3 & 11 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 7 & -6 \\ 8 & 23 & 12 & 15 \\ 21 & 13 & -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 9 & 3 & 0 \\ 1 & 23 & -2 & 20 \\ 33 & 10 & 10 & 4 \end{bmatrix}$$

Subtraction is done in the same way, by subtracting corresponding entries.

30. A NUMBER TIMES A MATRIX

To multiply a number r times a matrix A , just multiply each entry of A by r . The resulting matrix will have the same dimension as A .

$$3 \cdot \begin{bmatrix} 2 & 8 & -4 & 6 \\ -7 & 0 & -14 & 5 \\ 12 & -3 & 11 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 24 & -12 & 18 \\ -21 & 0 & -42 & 15 \\ 36 & -9 & 33 & 18 \end{bmatrix}$$

31. EQUALITY OF MATRICES

Two matrices A and B are equal if

1. they have the same dimensions and
2. their entries are identical to each other.

32. MULTIPLICATION OF MATRICES

We do **not** multiply matrices by simply multiplying entries, as in multiplication (or subtraction).

The definition is more complicated.

Dimension Check. In order to multiply A by B , you must verify that

the number of columns of A equals the number of rows of B

Put differently, the dimension of A must be $m \times p$, the dimension of B must be $p \times n$. The dimension of the product $A \cdot B$ is $m \times n$.

33. DIMENSIONS OF PRODUCT

$\dim(A)$	$\dim(B)$	AB defined?	$\dim(AB)$
2×3	2×3	No	Undefined
5×7	7×2	Yes	5×2
1×3	3×1	Yes	1×1
3×1	1×3	Yes	3×3

34. DEFINITION OF MULTIPLICATION

Once you have verified that the dimensions are correct, you can begin the process of actually multiplying the matrices $A \cdot B$.

Start with the first row of A and the first column of B . Simultaneously move across the row of A and the the column of B , multiply respective entries, and add up these products.

Repeat this process for every row of A and every column of B .

An example (or two) helps make this clear.

6	2	-7	4	1	
0	1	3	0	-1	
7	7	5	2	7	

Row 1 Column 1

6	2	-7	4		?
			0		
			2		

$$6 \cdot 4 + 2 \cdot 0 + -7 \cdot 2 = 24 + 0 + -14 = 10$$

Row 1 Column 2

6	2	-7		1	?
				-1	
				7	

$$6 \cdot 1 + 2 \cdot -1 + -7 \cdot 7 = 6 + -2 + -49 = -45$$

Row 2 Column 1

			4		
0	1	3	0		?
			2		

$$0 \cdot 4 + 1 \cdot 0 + 3 \cdot 2 = 0 + 0 + 6 = 6$$

Row 2 Column 2

				1	
0	1	3		-1	?
				7	

$$0 \cdot 1 + 1 \cdot -1 + 3 \cdot 7 = 0 + -1 + 21 = 20$$

Row 3 Column 1

			4		
			0		
7	7	5	2		?

$$7 \cdot 4 + 7 \cdot 0 + 5 \cdot 2 = 28 + 0 + 10 = 38$$

Row 3 Column 2

				1	
				-1	
7	7	5		7	?

$$7 \cdot 1 + 7 \cdot -1 + 5 \cdot 7 = 7 + -7 + 35 = 35$$

6	2	-7	4	1	10	-45
0	1	3	0	-1	6	20
7	7	5	2	7	38	35